AN IMPROVEMENT OF THE POINCARÉ-BIRKHOFF FIXED POINT THEOREM

By

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In 1912 Poincaré stated, and proved for some special cases, his Geometric Theorem. This theorem states that if h is an area preserving twist homeomorphism of the annulus $1 \leq r \leq 2$ onto itself, then h has at least two fixed points. In 1913 G.D. Birkhoff offered a proof of this result and in 1925 he stated a generalization in which the area preserving requirement is replaced by a topological condition, namely that no ring with $r = 1$ as part of its boundary is mapped onto a proper subset of itself by h or h^{-1}. This generalization, known as the Poincaré-Birkhoff Fixed Point Theorem, may be stated as follows: Let A be the annulus bounded by $r = 1$ and a simple closed curve c which lies in $r > 1$ and intersects each radial in exactly one point. If h is a twist homeomorphism of A onto the annulus h(A) where h(A) is bounded by $r = 1$ and the simple closed curve h(c) which lies in $r > 1$ and intersects each radial in exactly one point, then either there is some ring S with $r = 1$ for its inner boundary which is mapped onto a proper subset of itself by h or h^{-1}, or else h has at least two fixed points.

There has been some uneasiness and doubt about the correctness of the proofs given by Birkhoff and there recently has been considerable

effort to find proofs of these two theorems. In 1977 Morton Brown and W.D. Neumann gave a convincing proof of the area preserving theorem. There has also been some question as to whether the ring S constructed by Birkhoff in his proof of the generalization has a simple closed curve for its outer boundary. (Birkhoff's definition of ring does not require the boundary to be the union of two simple closed curves.)

The theorem proven in this dissertation is the following: Let A be the annulus bounded by $r = 1$ and the simple closed curve c which lies in $r > 1$ and intersects each radial in exactly one point. If h is a twist homeomorphism of A onto the annulus h(A) where h(A) is bounded by $r = 1$ and the simple closed curve h(c) which lies in $r > 1$ and intersects each radial in exactly one point, then either:

(1) There is a simple closed curve J which lies in the interior of A so that the annulus bounded by $r = 1$ and J is mapped onto a proper subset of itself by h,

(2) there is a simple closed curve J which lies in the interior of h(A) so that the annulus bounded by $r = 1$ and J is mapped onto a proper subset of itself by h^{-1}, or else,

(3) h has at least two fixed points.

There is a homeomorphism h so that the ring S given by Birkhoff's construction does not have a simple closed curve for its outer boundary. Thus the theorem stated is an improvement of the Poincaré-Birkhoff Fixed Point Theorem.

CHAPTER ONE
INTRODUCTION

History of the Theorem

In 1912 Poincaré enunciated, and proved for some special cases, his geometric theorem [15] which may be stated in the following manner:

> If A is the annulus $1 \leq r \leq 2$ and if g is an area preserving homeomorphism of A onto itself which moves points on $r = 1$ and $r = 2$ in opposite angular directions to their new positions on $r = 1$ and $r = 2$ respectively (that is to say, g is a twist homeomorphism of A onto itself), then g has at least two fixed points.

In 1913 G.D. Birkhoff offered the first proof of this theorem in his <u>Proof of Poincaré's Geometric Theorem</u> [3] and in 1925 he announced a more general version in <u>An Extension of Poincaré's Last Geometric Theorem</u> [4]. This generalization, which will be called the Poincaré-Birkhoff fixed point theorem in the sequel (although in the literature this name is used for either theorem), may be stated as follows:

> Let A be the annulus in the plane bounded by $r = 1$ and a simple closed curve γ which lies in $r > 1$ and intersects each radial in exactly one point. If g is a homeomorphism of A onto the annulus g(A) where g(A) is bounded by $r = 1$ and the simple closed curve $g(\gamma)$ which lies in $r > 1$ and intersects each radial in exactly one point, and if, further, g moves points on $r = 1$ and γ

in opposite angular directions to their new positions on $r = 1$ and $g(\gamma)$ respectively (that is to say, g is a twist homeomorphism of A onto $g(A)$), then either there is a ring S which has $r = 1$ as its inner boundary and which is mapped onto a proper subset of itself by g or g^{-1}, or else g has at least two fixed points.

A few comments about the statements of these two theorems are necessary. First, the requirement that g move points on the two components of the boundary of A in opposite angular directions is ambiguous and needs a rigorous formulation. One way this may be done is by requiring that some lift of g to a homeomorphism of the universal cover $\tilde{A}$ of A move points on the two components of the boundary of $\tilde{A}$ in opposite directions. Second, Birkhoff defined a ring to be the region bounded by two continua, C_1 and C_2, so that the first is a subset of the union of the second and the bounded component of its complement, where each of C_1 and C_2 is the common boundary of a bounded, connected, simply connected open set and the complement of its closure. Note that C_1 and C_2 were not required to be simple closed curves. Finally, the most important difference between the Poincaré-Birkhoff fixed point theorem and Poincaré's geometric theorem is the replacement of the area preserving condition by the more general requirement that no ring be mapped onto a proper subset of itself by g or g^{-1}.

The correctness of Birkhoff's arguments for these two theorems has been doubted, especially the correctness of his argument for the existence of the second fixed point, see for example [17]. His argument for the second fixed point in his 1913 paper is definitely in error, but in his argument in [4] (which is summarized in [5]) to show the more general Poincaré-Birkhoff fixed point theorem he avoids the obvious error made in his earlier paper. Alternative

proofs for the existence of the first fixed point in the Poincaré-Birkhoff fixed point theorem have been offered by, for example, Kerékjarto in 1928 [11] and Barrar in 1967 [2]. Morton Brown and W.D. Neumann have recently given in [6] a convincing proof, along the lines of Birkhoff's original arguments, of the area preserving theorem of Poincaré. In [9] and [10] H. Jacobowitz presented an argument for an area preserving theorem slightly more general than Poincaré's. A number of mathematicians have given arguments for the analogous theorem for area preserving flows, see [1] and [8] for example. No one has yet offered a convincing argument for the Poincaré-Birkhoff fixed point theorem, although Morton Brown and W.D. Neumann have indicated in [6] that a proof of this theorem could be had by applying their careful technique to the ideas in Birkhoff's 1925 paper.

There has also been some question as to whether the ring S in the conclusion of the Poincaré-Birkhoff fixed point theorem has the property that the boundary of $S \cup \{(r,\theta): r < 1\}$ is a simple closed curve. Some authors, for example van der Walt in [18], have misquoted Birkhoff's result, requiring the boundary of $S \cup \{(r,\theta): r < 1\}$ to be a simple closed curve. In the third chapter an example of a twist homeomorphism g of the annulus $1 \leq r \leq 3\frac{1}{20}$ onto itself will be given for which the ring S, as constructed by Birkhoff in his argument in [4] for the Poincaré-Birkhoff fixed point theorem, does not have this property. Moreover, it will be shown that for this ring S there is no simple closed curve in $\overline{S} \setminus g(S)$ which separates the boundary components of A. So the application of careful technique to Birkhoff's ideas for the proof of his 1925 theorem would not result in the proof of a theorem in which the ring S is required to have two simple closed curves for its boundary.

In chapter two a rigorous formulation and proof of the following improvement of the Poincaré-Birkhoff fixed point theorem will be given:

Let A be the annulus bounded by $r = 1$ and a simple closed curve γ which lies in $r > 1$ and intersects each radial in exactly one point. If g is a twist homeomorphism of A onto the annulus g(A) where g(A) is bounded by $r = 1$ and a simple closed $g(\gamma)$ which lies in $r > 1$ and intersects each radial in exactly one point then either (1) there is a simple closed curve θ in the interior of A so that the annulus bounded by $r = 1$ and θ is mapped onto a proper subset of itself by g, or (2) there is a simple closed curve θ in the interior of g(A) so that the annulus bounded by $r = 1$ and θ is mapped onto a proper subset of itself by g^{-1}, or else (3) g has at least two distinct fixed points.

This theorem clearly implies the Poincaré-Birkhoff fixed point theorem which Birkhoff enunciated in his 1925 paper, and is actually an improvement as the previous paragraph indicates.

Some Examples of Twist Homeomorphisms of the Annulus

In this section we give some simple examples of twist homeomorphisms of the annulus $A = \{(r,\theta): 1 \leq r \leq 2\}$ onto itself.

Let $g_1: A \twoheadrightarrow A$ be the radial homeomorphism given by $g_1(r,\theta) = (\sqrt{r - 1} + 1, \theta)$ and let $g_2: A \twoheadrightarrow A$ be the angular homeomorphism given by $g_2(r,\theta) = (r, \theta + \frac{\pi}{2} r - \frac{3\pi}{4})$. On each radial g_1 fixes the intersection of that radial with $r = 1$ and $r = 2$ and moves every other point outward. On each circle $r = \lambda$, $1 \leq \lambda \leq 2$, g_2 is a rotation of $\frac{\pi}{2} \lambda - \frac{3\pi}{4}$; for example g_2 rotates $r = 1$ through an angle of $-\frac{\pi}{4}$, $r = 1\frac{1}{2}$ through an

angle of 0 and $r = 2$ through an angle of $\frac{\pi}{4}$. The map $f_1 = g_2 \cdot g_1$ is a twist homeomorphism with no fixed points. Note that for every λ, $1 < \lambda < 2$, the annulus $1 \le r \le \lambda$ is mapped onto a proper subset of itself by f_1^{-1}.

For a twist homeomorphism with exactly one fixed point let $g_3 \colon A \twoheadrightarrow A$ be the radial homeomorphism given by $g_3(r,\theta) = (r + (\sqrt{r-1} + 1 - r)\sin \frac{\theta}{2}, \theta)$, $0 \le \theta < 2\pi$, and let $g_2 \colon A \twoheadrightarrow A$ be the angular homeomorphism given by $g_2(r,\theta) = (r, \theta + \frac{\pi}{2} r - \frac{3\pi}{4})$, as in the previous example. Note that g_3 is a homeomorphism of each radial onto itself which fixes $r = 1$ and $r = 2$ and g_3 is the identity on the radial $\theta = 0$, on other radials g_3 moves points outward. The map $f_2 = g_2 \cdot g_3$ is a twist homeomorphism with exactly one fixed point, $(1\frac{1}{2}, 0)$. Every annulus $1 \le r < \lambda$, for $1 < \lambda < 2$, is mapped onto a proper subset of itself by f_2^{-1}. Each circle $r = \lambda$, $1 < \lambda < 2$, intersects its image under f_2^{-1} in exactly one point.

For a twist homeomorphism with exactly two fixed points let $g_4 \colon A \twoheadrightarrow A$ be the radial homeomorphism given by $g_4(r,\theta) = (r + (\sqrt{r-1} + 1 - r)\sin\theta, \theta)$, $0 \le \theta < 2\pi$, and let g_2 be as before. Note that g_4 is a homeomorphism of each radial onto itself which fixes $r = 1$ and $r = 2$, and g_1 is the identity on $\theta = 0$ and $\theta = \pi$; for $0 < \theta < \pi$, g_4 moves points outward and on $\pi < \theta < 2\pi$, g_4 moves points inward. The map $f_3 = g_2 \cdot g_4$ is a twist homeomorphism of the annulus onto itself with exactly two fixed points $(1\frac{1}{2}, 0)$ and $(1\frac{1}{2}, \pi)$.

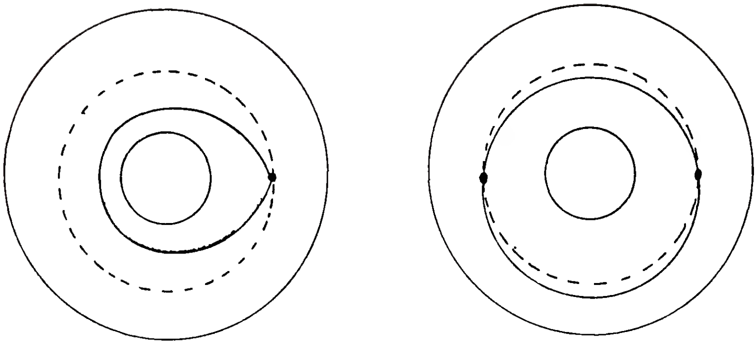

Figure 1. The images of $r = 1\frac{1}{2}$ under f_2^{-1} and f_3^{-1}.

Examples of twist homeomorphisms which are also flows are easy to illustrate. The flow pictured in Figure 2 can be easily modified to create one, two, or more fixed points. The flow pictured in Figure 3 with exactly two fixed points can be made to be area preserving.

Preliminaries

If $P = (p_1,p_2)$ and $Q = (q_1,q_2)$ are points (in Cartesian coordinates) in the plane E^2 then $P + Q = (p_1 + q_1, p_2 + q_2)$. A simple curve α is a one-to-one continuous function from $[0,1]$ into the plane and the symbol α will often be used for the image $\alpha([0,1])$ as well as the map. A simple closed curve α is a one-to-one continuous map of the unit circle into the plane and the symbol α will often by used to denote the image as well as the map. Between any two points p and q on the simple closed

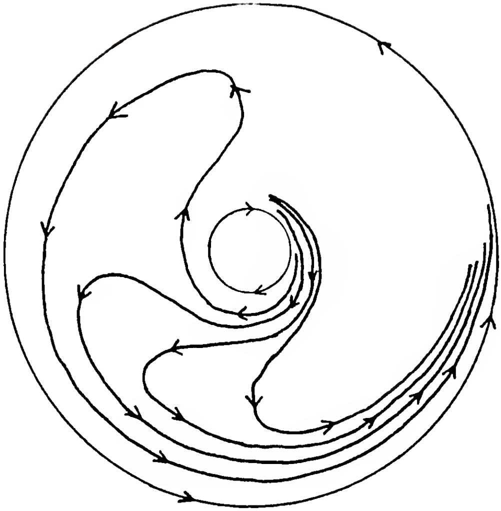

Figure 2. A fixed point free flow.

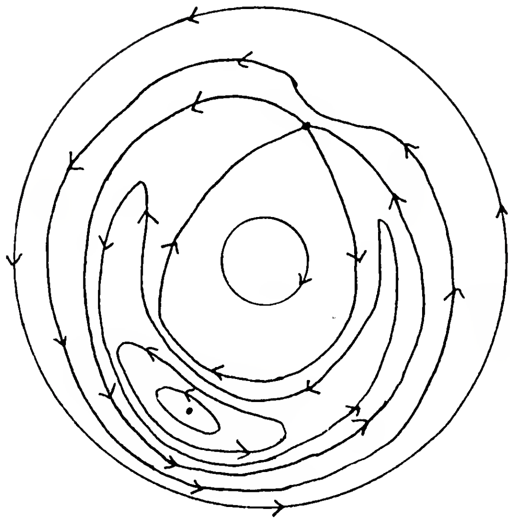

Figure 3. A flow with exactly 2 fixed points.

curve α there are two arcs, $[p,q]_{cc}$ and $[p,q]_c$ where cc denotes counter-clockwise and c denotes clockwise. For a simple closed curve α the interior of α, or intα, denotes the bounded component of the complement of α and the exterior of α, or extα, denotes the unbounded component of the complement of α. For other sets interior has its usual meaning. For a set A, A^c denotes the complement of A. For sets A and B, $A \setminus B = A \cap B^c$. The composition of two functions f and g will be denoted by fg, $f \cdot g$, or $f(g(x))$. The symbols $\|P\|$, $d(A,B)$, $\mathrm{diam}(A)$, LUB, Max, Min, $\lim\limits_{x\to a}$ sup, and $\lim\limits_{x\to a}$ inf will have their usual meanings.

The notations, definitions, and theorems concerning prime ends used in the section constructing δ-chains are as in the chapter on prime ends in [7].

The following definition of index and lemmas 1.1 through 1.6 appear in [6]. Suppose $X \subset E^2$ and h is a homeomorphism of X into E^2 and suppose $C: [a,b] \to X$ is a continuous function so that h has no fixed points on $C([a,b])$. Let $\bar{h}(P) = D(P,h(P)) = (h(P) - P) \;/\; \|h(P) - P\|$ and let $\Pi: R \to S^1$ by $\Pi(x) = (\cos x, \sin x)$. Then there is a map $\tilde{C}$ from $[a,b]$ into R which satisfies $\Pi\tilde{C} = \bar{h}$, that is, which makes the following diagram commute:

$$\begin{tikzcd} & R \arrow[d, "\Pi"] \\ {[a,b]} \arrow[ur, dashed, "\tilde{C}"] \arrow[r, "\bar{h}"'] & S^1 \end{tikzcd}$$

Finally, define $\mathrm{Ind}_h C$ to be $(\tilde{C}(b) - \tilde{C}(a))/2\pi$. The following lemmas will be freely used in the last three sections of the proof of the theorem.

LEMMA 1.1. If C_t, $0 \leq t \leq 1$, is a continuous family of curves, none of which contains a fixed point of h then $\mathrm{Ind}_h C_t$ is a continuous function of t.

LEMMA 1.2. If h_t, $0 \leq t \leq 1$, is a continuous family of homeomorphisms, none of which has a fixed point on C, then $\mathrm{Ind}_{h_t} C$ is continuous function of t.

LEMMA 1.3. If C: $[a,b] \to X$ then $\mathrm{Ind}_h C = \theta/2\pi$ where θ is the angle between $D(C(a), hC(a))$ and $D(C(b), hC(b))$.

LEMMA 1.4. Suppose C_1: $[a,b] \to X$ and C_2: $[b,c] \to X$, and $C = C_1 \cup C_2$: $[a,c] \to X$ are continuous functions then $\mathrm{Ind}_h C = \mathrm{Ind}_h C_1 + \mathrm{Inc}_h C_2$.

LEMMA 1.5. Suppose C: $[a,b] \to X$ and $-C$: $[a,b] \to X$ is given by $-C(t) = C(b + a - t)$ then $\mathrm{Ind}_h -C = -\mathrm{Ind}_h C$.

LEMMA 1.6. Suppose C: $[a,b] \to X$ then $\mathrm{Ind}_h C = \mathrm{Ind}_{h^{-1}} h(C)$.

CHAPTER TWO

PROOF OF THE THEOREM

A Precise Statement of the Theorem

Let A be the annulus in the plane which is bounded by $r = 1$ and a simple closed curve γ which lies in $r > 1$ and intersects each radial in exactly one point. Suppose g is a homeomorphism of the annulus A onto the annulus $g(A)$ where $g(A)$ is bounded by $r = 1$ and the simple closed curve $g(\gamma)$ which lies in $r > 1$ and intersects each radial in exactly one point, and further suppose that g has no fixed points on $r = 1$ or γ. Now let Π map $\{(x,y): y \geq 0\}$ onto $\{(r,\theta): r \geq 1\}$ by $\Pi(x,y) = ((y + 1)\cos x, (y + 1)\sin x)$ and let $\tilde{A} = \Pi^{-1}(A)$. There is a homeomorphism $\tilde{g}$ of $\tilde{A}$ into the plane so that $g \cdot \Pi = \Pi \cdot \tilde{g}$, that is to say, so that $\tilde{g}$ makes the following diagram commute:

$$\begin{tikzcd}
\tilde{A} \arrow[r, "\tilde{g}"] \arrow[d, "\Pi"] & \tilde{g}(\tilde{A}) \arrow[d, "\Pi"] \\
A \arrow[r, "g"] & g(A)
\end{tikzcd}$$

Since $\Pi^{-1}(\gamma)$ is a one-to-one, continuous image of the real line which is contained in $y > 0$ and intersects each vertical line in exactly one point there is a continuous function $\Gamma: R \to R$ so that $\Pi^{-1}(\gamma) = \{(x,y): y = \Gamma(x)\}$. Also there is a continuous function $\Gamma_1: R \to R$ so that $\Pi^{-1}(g(\gamma)) =$

$g\{(x,y): y = \Gamma(x)\} = \{(x,y): y = \Gamma_1(x)\}$. The homeomorphism g from A onto $g(A)$ is a <u>twist homeomorphism</u> if $\tilde{g}$ can be chosen so that $\tilde{g}$ moves points of $y = 0$ and $y = \Gamma(x)$ in opposite directions, that is, so that either for every x the x-coordinate of $\tilde{g}(x,0)$ is greater than x and the x-coordinate of $\tilde{g}(x,\Gamma(x))$ is less than x or else for every x the x-coordinate of $\tilde{g}(x,0)$ is less than x and the x-coordinate of $\tilde{g}(x,\Gamma(x))$ is greater than x. In the sequel assume, without loss of generality, that the first holds, that is,

$$\tilde{g}(x,0) = (x',0) \text{ with } x' > x, \text{ and}$$
$$\tilde{g}(x,\Gamma(x)) = (x'',y) \text{ with } x'' < x.$$

Now if g is a twist homeomorphism from A onto $g(A)$ then g^{-1} is a twist homeomorphism from $g(A)$ onto A and the homeomorphism $(\tilde{g})^{-1}$ from $\tilde{g}(\tilde{A})$ onto $\tilde{A}$ satisfies $g^{-1}\Pi = \Pi(\tilde{g})^{-1}$. Note that $(\tilde{g})^{-1}$ satisfies

$$(\tilde{g})^{-1}(x,0) = (x',0) \text{ with } x' < x, \text{ and}$$
$$(\tilde{g})^{-1}(x,\Gamma_1(x)) = (x'',y) \text{ with } x'' > x.$$

The main theorem can now be stated.

MAIN THEOREM. If g is a twist homeomorphism of the annulus A onto $g(A)$ then either:

(1) There is a simple closed curve θ in the interior of A so that the annulus bounded by θ and $r = 1$ is mapped onto a proper subset of itself by g,

(2) there is a simple closed curve θ in the interior of $g(A)$ so that the annulus bounded by θ and $r = 1$ is mapped onto a proper subset of itself by g^{-1}, or else,

(3) g has at least two distinct fixed points.

In order to prove this we will assume that neither is the case, that is that there is no such simple closed curve θ and that g has at most one fixed point F.

In the sequel it will be convenient to have a homeomorphism of the whole plane onto itself instead of a homeomorphism from $\tilde{A}$ onto $\tilde{g}(\tilde{A})$. So define the homeomorphism $h: E^2 \twoheadrightarrow E^2$ by:

$$h(x,y) = \begin{cases} \tilde{g}(x,y) & \text{if } 0 \le y \le \Gamma(x) \\ \tilde{g}(x,0) + (0,y) & \text{if } y < 0 \\ \tilde{g}(x,\Gamma(x)) + (0,y-\Gamma(x)) & \text{if } y > \Gamma(x). \end{cases}$$

The following three properties of h are immediate from its definition:

(1) $g \cdot \Pi = \Pi \cdot h|_{\tilde{A}}$,

(2) $h(x+2\pi,y) = h(x,y) + (2\pi,0)$ for all (x,y) in E^2, and

(3) $h(x,0) = (x',0)$ for some $x' > x$,

$h(x,\Gamma(x)) = (x'',y)$ for some $x'' < x$.

Further, if $(x,y) \in \tilde{A}$ and $h(x,y) = (x+2n\pi,y)$ for some integer n, then $\Pi(x,y)$ is a fixed point of g since $g \cdot \Pi(x,y) = \Pi \cdot h|_{\tilde{A}}(x,y) = \Pi(x+2n\pi,y) = \Pi(x,y)$. If $y < 0$ and $h(x,y) = (x+2n\pi,y)$, then $h(x,0) = (x+2n\pi,0)$ so that by the previous argument $\Pi(x,0)$ is a fixed point of g on $r = 1$, contradicting the fact that g is fixed point free on $r = 1$. If $y > \Gamma(x)$ and $h(x,y) = (x+2n\pi,y)$ then $h(x,\Gamma(x)) = (x+2n\pi,\Gamma(x))$ so that $\Pi(x,\Gamma(x))$ is a fixed point of g on γ contradicting the fact that g is fixed point free on γ. So h has the additional property:

(4) $h(x,y) = (x+2n\pi,y)$ for some integer n if only if $\Pi(x,y)$ is a fixed point of g.

If g is fixed point free on A, then h is fixed point free on E^2 and, in fact, $h(x,y) \neq (x+2n\pi,y)$ for all (x,y) in E^2 and all integers n. If g

has exactly one fixed point in A, then there is some $(\bar{x},\bar{y})$ with $0 < y < \Gamma(\bar{x})$ so that $g\cdot\Pi(\bar{x},\bar{y}) = \Pi(\bar{x},\bar{y})$. In this case there is some integer N so that every point (x,y) in $\tilde{F} = \{(\bar{x}+2k\pi,\bar{y})$: k is an integer$\}$ satisfies $h(x,y) = (x+2N\pi,y)$. So either h has $\tilde{F}$ as its set of fixed points, or else h has no fixed points and the only points (x,y) satisfying $h(x,y) = (x+2n\pi,y)$ for any integer n are those in $\tilde{F}$.

In the section on the construction of the auxiliary homeomorphism it will be necessary to have the homeomorphism g extended to a slightly larger set. We can extend g in a natural way to all of E^2 by requiring, where $g(r,\theta) = (g_1(r,\theta),\ g_2(r,\theta))$ and $\gamma = \{(r,\theta): r = \gamma(\theta)\}$,

$$g(r,\theta) = \begin{cases} g(r,\theta) & \text{if } (r,\theta) \in A \\ (r,\ g_2(1,\theta)) & \text{if } r < 1 \\ (g_1(\gamma(\theta),\theta) + r - \gamma(\theta),\ g_2(\gamma(\theta),\theta)) & \text{if } r > \gamma(\theta). \end{cases}$$

If Π maps $\{(x,y): y > -1\}$ onto $\{(r,\theta): r > 0\}$ by $\Pi(x,y) = ((y+1)\cos x,\ (y+1)\sin x)$ then the extended function g satisfies

$$g\cdot\Pi = \Pi\cdot h\big|_{y > -1} \;.$$

Overview of the Proof

In the first section of the proof a continuous nonnegative function $\delta: A \to R$ is chosen to satisfy these requirements:

(1) $\overline{N(P,\delta(P))} \cap \overline{g(N(P,\delta(P))} = \phi$ if $P \neq F$;

(2) $\overline{N(P,\delta(P))} \cap \overline{g^{-1}N(P,\delta(P)))} = \phi$ if $P \neq F$; and

(3) $\delta(P) = 0$ if and only if $P = F$.

(Note here that this differs from Birkhoff's construction in that his δ

is a function of the θ-coordinate only.) A δ-chain is then defined to be a finite sequence of points $P_0, P_1, \ldots, P_N$ from $r \geq 1$, with P_0 in $r = 1$, $P_{k+1} = g(P_k) + V_k$ for some point V_k where $\| V_k \| < \delta(g(P_k))$ and $P_k \in A$ for $0 \leq k < N$. If $P_N \not\in A$ then $P_0, P_1, \ldots, P_N$ is said to be terminating. Let $M_n = \{P: P = P_n$ for some δ-chain $P_0, \ldots, P_n\}$, and note by choice of δ, $F \not\in M_n$. If there were no terminating δ-chains, then $M_n \subset \{(r,\theta): 1 \leq r < \gamma(\theta)\}$ for each n. We then let $M = \bigcup_{n=1}^{\infty} M_n$ and let S be the complement of the closure of the unbounded component of the complement of the closure of M, minus $r < 1$. The ring S has the property that $g(S) \subset S$ and that there is a simple closed curve θ in $\overline{S} \setminus g(S)$ which separates the boundary components of A. Hence $r = 1$ and θ bound an annulus which is mapped onto a proper subset of itself by g, contradicting the standing assumption that there is no such annulus. Hence there is a terminating δ-chain for g. A terminating δ-chain $P_0, P_1, \ldots, P_N$ is then chosen so that N is minimal, and so that $N(g(P_{N-1}), \|V_{N-1}\|) \cap \gamma = \phi$. Let $\delta_k = \|V_k\|$. This δ-chain $P_0, P_1, \ldots, P_N$ and the numbers δ_k will remain fixed throughout the remainder of the argument.

In the second section of the proof an auxiliary homeomorphism $T: \{(r,\theta): r \geq 1 - \eta\} \twoheadrightarrow \{(r,\theta): r \geq 1\}$ is constructed, for a certain positive number η so that $T(g(P_k)) = P_{k+1}$ and so that $T \cdot g$ is homotopic to g via a homotopy which has at each level exactly the same fixed points as g. Essentially T is constructed as follows: First we find a collection of pairwise disjoint arcs α_k from $g(P_k)$ to P_k where $\alpha_k \subset N(g(P_k), \delta(g(P_k)))$. It is possible to pick such arcs α_k satisfying the last requirement because of the minimality properties of $P_0, P_1, \ldots, P_N$ and because the sets $N(g(P_k), \delta(g(P_k)))$ are disks. Then we find a collection of pairwise disjoint topological disks D_k

so that $\alpha_k \subset \mathrm{int} D_k \subset N(g(P_k), \delta(g(P_k)))$. Inside each disk D_k, $g(P_k)$ can now be mapped to P_{k+1} by some homeomorphism T which leaves the boundary of each disk D_k fixed, while outside $\bigcup_{k=0}^{N-1} D_k$ T is the identity. Then T is fixed up for the special case $g(P_0)$ to P_1, and close to $r = 1$. Since T only moves points inside the disjoint disks $D_k \subset N(g(P_k), \delta(g(P_k)))$ and points close to $r = 1$, it does not create any new fixed points, nor obliterate any existing fixed point, of g (by choice of δ and η). The homotopy between g and $T \cdot g$ is lifted to a homotopy between $H_0 = h$ and a homeomorphism H_1.

In the third section of the proof a simple curve C^* is constructed as follows: Let $\hat{P}_0 \in \Pi^{-1}(P_0)$ and for each k let $\hat{P}_k = H_1^k(\hat{P}_0)$. Let C_0 be the straight line segment from $\hat{P}_{-1}$ on $y = -\eta$ to $\hat{P}_0$ on $y = 0$ and let $C^* = \bigcup_{k=0}^{N} H_1^k(C_0)$. The curve C^* is then shown to be a simple curve from $\hat{P}_{-1}$ on $y = -\eta$ to $\hat{P}_N$ on $y = \Gamma(x)$ which contains no point of $\Pi^{-1}(F)$. If θ_1^t is the angle from the vector $\overrightarrow{\hat{P}_{-1}, \hat{P}_{-1} + (1,0)}$ to $\overrightarrow{\hat{P}_{-1}, H_t(\hat{P}_{-1})}$ and θ_2^t is the angle from the vector $\overrightarrow{\hat{P}_N, \hat{P}_N + (1,0)}$ to $\overrightarrow{\hat{P}_N, H_t(\hat{P}_N)}$ then $\mathrm{Ind}_{H_t}(C^*) = (\theta_2^t - \theta_1^t)/2\pi \pmod 1$; by choice of δ and η, $0 \le \theta_1^t \le \pi/4$ and $\pi/2 < \theta_2^t < 3\pi/2$ so that $\frac{1}{8} < \mathrm{Ind}_{H_t} C^* < \frac{3}{4} \pmod 1$. In this section we also show that if W is a point in $y \le 0$ and B is a point in $y \ge \Gamma(x)$, then $\mathrm{Ind}_{H_t} C$ is the same for every simple arc C from W to B which misses the fixed points of h if any. (If h has fixed points, then the set of fixed points of h is $\Pi^{-1}(F)$.)

For this particular homeomorphism H_1 and this particular curve C^* homotopies are constructed in the fourth section of the proof to show that $\mathrm{Ind}_{H_1} C^* = (\theta_2^1 - \theta_1^1)/2\pi$. Since for all t, $0 \le t \le 1$, $\mathrm{Ind}_{H_t} C^* = (\theta_2^t - \theta_1^t)/2\pi \pmod 1$, using the continuity property of the index we have $\mathrm{Ind}_h C^* = (\theta_2^0 - \theta_1^0)/2\pi$. Hence we have $\frac{1}{4} < \mathrm{Ind}_h C^* < \frac{3}{4}$.

Now an analogous argument could be made for the twist homeomorphism g^{-1}, using h^{-1} for the lift of g^{-1}, with the only essential differences being that g^{-1} maps $g(A)$ onto A and that h^{-1} moves points on $y = 0$ to points on $y = 0$ with smaller x-coordinates and h^{-1} moves points on $y = \Gamma_1(x)$ to points on $y = \Gamma(x)$ with larger x-coordinates. Hence the end result of such an argument would be that there exists a simple closed curve B from some point in $y \leq 0$ to some point in $y = \Gamma_1(x)$ so that $-\frac{3}{4} < \mathrm{Ind}_{h^{-1}} B < -\frac{1}{4}$.

In the last section of the proof we obtain the desired contradiction in this manner: The curve $B^* = h^{-1}(B)$ from some point Z_2 in $y \leq 0$ to some point W_2 in $y = \Gamma(x)$ satisfies $-\frac{3}{4} < \mathrm{Ind}_h B^* < -\frac{1}{4}$ since $\mathrm{Ind}_h h^{-1}(B) = \mathrm{Ind}_{h^{-1}} B$. Translation by multiples of 2π does not change the index here, so assume C^* and B^* are disjoint. Suppose C^* is from Z_1 in $y \leq 0$ to W_1 in $y = \Gamma(x)$. Let α_1 be a simple arc from Z_1 to Z_2 in $y \leq 0$ and let α_2 be that part of $y = \Gamma(x)$ from W_2 to W_1. Then $\mathrm{Ind}_h\, \alpha_1 B^* \alpha_2 = \mathrm{Ind}_h C^*$, but $\frac{1}{4} < \mathrm{Ind}_h C^* < \frac{3}{4}$, and since $\mathrm{Ind}_h\, \alpha_1 B^* \alpha_2 = \mathrm{Ind}_h\, \alpha_1 + \mathrm{Ind}_h B^* + \mathrm{Ind}_h \alpha_2$, where $\mathrm{Ind}_h\, \alpha_1 = 0$, $-\frac{1}{2} < \mathrm{Ind}_h\, \alpha_2 < \frac{1}{2}$ and $-\frac{3}{4} < \mathrm{Ind}_h B^* < -\frac{1}{4}$, $-\frac{5}{4} < \mathrm{Ind}_h\, \alpha_1 B^* \alpha_2 < \frac{1}{4}$. Since we have obtained a contradiction from the assumptions that there is no simple closed curve θ so that the annulus bounded by $r = 1$ and θ is mapped onto a proper subset of itself by g or by g^{-1} and g has at most one fixed point, the theorem is proven.

Constructing δ-Chains

In this section we show that the assumptions that there is no simple closed curve θ so that the annulus determined by $r = 1$ and θ is mapped onto a proper subset of itself by either g or g^{-1}, and that g has at most one fixed point, imply the existence of, for each of g and g^{-1}, a finite sequence of points with certain useful properties which will be crucial in the remainder of the argument.

First, there is a positive number μ_1 so that $d(P,h(P)) > \mu_1$ for all points P in $y = 0$ or $y = \Gamma(x)$, since h is periodic in the sense that $h(x+2\pi,y) = h(x,y)$ and has no fixed points on $y = 0$ or $y = \Gamma(x)$. Further, there is a positive number ζ so that $\zeta < \min\{|x-x'|: (x',y') = h(x,\Gamma(x))\}$. Also, there is a positive number $\mu_2 < \frac{1}{3}\min(\mu_1,\zeta)$ so that if $\mathrm{diam}(B) < \mu_2$ then $\mathrm{diam}(h(B)) < \frac{1}{3}\min(\mu_1,\zeta)$ since h is a periodic homeomorphism, and since on $E^2 \setminus \tilde{A}$, h is determined by its action on $y = 0$ and $y = \Gamma(x)$. Finally, there is a positive number μ_3 so that for all points P in A, if $\varepsilon \le \mu_3$, then each component of $\Pi^{-1}(N(P,\varepsilon))$ has diameter less than μ_2. Note that if $\Pi(\tilde{Q}) = Q$ is on $r = 1$ and O is the component of $\Pi^{-1}(N(Q,\mu_3))$ which contains $\tilde{Q}$ then every point of $h(O)$ is to the right of $\tilde{Q}$, and that if $\Pi(\tilde{Q}) = Q$ is on γ and O is the component of $\Pi^{-1}(N(Q,\mu_3)$ which contains $\tilde{Q}$ then every point of $h(O)$ is to the left of $\tilde{Q}$.

DEFINITION 2.1. If g does not have any fixed points in A then there is a positive number δ so that the following hold:

(1) $\overline{N(P,\delta)} \cap g\,\overline{N(P,\delta)} = \phi$,

(2) $\overline{N(P,\delta)} \cap g^{-1}\,\overline{N(P,\delta)} = \phi$, and

(3) $\delta < \mu_3$.

Define $\delta\colon A \to R$ to be the constant map given by $\delta(P) = \delta$ for all P in A.

If g has exactly one fixed point F in A (F must lie in $\mathrm{int}A$), then there is a continuous, nonnegative function $\delta\colon A \to R$ with the following properties:

(1) $\delta(P) = 0$ if and only if $P = F$,

(2) $\delta(P) < d(F,P)$ if $P \neq F$,

(3) $\overline{N(P,\delta(P))} \cap g\,\overline{N(P,\delta(P))} = \phi$ if $P \neq F$,

(4) $\overline{N(P,\delta(P)} \cap g^{-1}\,\overline{N(P,\delta(P))} = \phi$ if $P \neq F$, and

(5) $\delta(P) < \mu_3$.

DEFINITION 2.2. A δ-chain is a finite sequence $P_0, P_1, \ldots, P_N$ of points in E^2 with these properties:

(1) $P_0 \in \{(r,\theta)\colon r = 1\}$,

(2) $P_k \in A$ for $0 \le k < N$,

(3) $P_N \in \{(r,\theta)\colon r \ge 1\}$, and

(4) $P_{k+1} = g(P_k) + V_k$ where $V_k \in E^2$ and $\|V_k\| < \delta(g(P_k))$, for $0 \le k < N$.

A δ-chain $P_0, P_1, \ldots, P_N$ is said to be <u>terminating</u> if $P_N \in \gamma \cup \mathrm{ext}\gamma$.

LEMMA 2.3. If there is no terminating δ-chain then there is a subset S of A with these properties:

(1) $r = 1$ is a subset of S,

(2) $S \setminus \{(r,\theta)\colon r = 1\}$ is open,

(3) $E^2 \setminus \bar{S}$ has exactly two components, and

(4) $g(S) \subseteq S$.

PROOF. Suppose there is no terminating δ-chain. For each nonnegative integer n let $M_n = \{P \in A\colon$ there is some δ-chain $P_0, P_1, P_2, \ldots, P_n$ with $P = P_n\}$. Then, for example, $M_0 = \{(r,\theta)\colon r = 1\}$ and $M_1 = \bigcup_{0 \le \theta < 2\pi} N((1,\theta),\delta(1,\theta)) \cap A = \{(r,\theta)\colon 1 \le r < 1 + \delta(1,\theta)\}$. For each nonnegative integer n, if P_n

is the n+1st element of the δ-chain $P_0, P_1, \ldots, P_n$ then P_n is the n+2nd element of the δ-chain $g^{-1}(P_0), P_0, P_1, \ldots, P_n$, so $P_n \in M_{n+1}$. Therefore, $M_n \subseteq M_{n+1}$ for all n. Suppose $P \in g(M_n)$, so that $g^{-1}(P) \in M_n$, then there is a δ-chain $P_0, P_1, \ldots, P_n$ with $P_n = g^{-1}(P)$. So $P_0, P_1, \ldots, P_n, g(P_n) + (0,0)$ is a δ-chain whose n+2nd element is P, hence $P \in M_{n+1}$. Therefore $g(M_n) \subseteq M_{n+1}$ for every n. Since, for each $n \geq 1, M_n = \bigcup_{P \in g(M_{n-1})} (N(P,\delta(P)) \cap A)$, M_n is connected and $M_n \setminus \{(r,\theta): r = 1\}$ is open.

Now let $M = \bigcup_{k=0}^{\infty} M_k$. Since there are no terminating δ-chains $M_n \subseteq A$ for each n, so that $M \subseteq A$. Since $g(M_k) \subseteq M_{k+1}$ and since $M_k \subseteq M_{k+1}$, $g(\bigcup_{k=0}^{\infty} M_k) \subseteq \bigcup_{k=0}^{\infty} M_{k+1} = M$, so that $g(M) \subseteq M$. Since $M \setminus \{(r,\theta): r = 1\}$ is the union of the open sets $M_k \setminus \{(r,\theta): r = 1\}$, it is open. Since each M_k is connected and $M_k \subseteq M_{k+1}$, M is connected. Since $\{(r,\theta): r = 1\} \subseteq M \subseteq A$, $E^2 \setminus \overline{M}$ has one component which is $\{(r,\theta): r < 1\}$ and exactly one unbounded component, which contains extγ. Thus M has all the properties desired of S except (3).

Let B be the union of all the bounded components of $E^2 \setminus \overline{M}$ except $\{(r,\theta): r < 1\}$, if any. Finally let $S = \text{int}(\overline{M} \cup B) \cup \{(r,\theta): r = 1\}$. Now, clearly, $\{(r,\theta): r = 1\} \subset S$, $S \setminus \{(r,\theta): r = 1\}$ is open, S is connected, and $E^2 \setminus S$ has exactly two components, one being $r < 1$ and the other containing $\gamma \cup \text{ext}\gamma$. Now, since $g(\overline{M}) \subseteq \overline{M}$, and since g maps points in the unbounded component of $E^2 \setminus \overline{M}$ in A to points in the unbounded component of $E^2 \setminus g(\overline{M})$ in g(A), the unbounded component of $E^2 \setminus \overline{M}$ is a subset of the unbounded component of $E^2 \setminus g(\overline{M})$, hence $g(S) \subseteq S$. □

Since S contains $M_1 = \{(r,\theta): 1 \leq r < 1 + \delta(1,\theta)\}$ the boundary of S has exactly two components; let ∂S denote that component which is not $r = 1$. Otherwise ∂B for a set B has its usual

meaning. Note that the fixed point F, if there is one, does not belong to any of the sets M_n, thus does not belong to M. However, F may belong to S or to ∂S.

LEMMA 2.4. Suppose g is fixed point free and δ is defined as in the first part of definition 2.1. If no δ-chain terminates, then there is a simple closed curve θ in intA with $r = 1$ in its interior and with $g(\theta) \subset \text{int}\theta$ (so that the annulus bounded by $r = 1$ and θ is mapped onto a proper subset of itself by g).

PROOF. Construct the set S as in lemma 2.3. Let $P \in \partial S$; since P is a limit point of M and since h is continuous there is a point Q in M so that $d(g(P),g(Q)) < \delta/4$. There is an integer n so that $Q \in M_n$, hence $N(g(Q),\delta) \cap A \subset M_{n+1} \subset S$. So, $N(g(P),\delta/4) \subset N(g(Q),\delta)$ so that $g(P) \in \text{int}S$. Therefore $g(\partial S)$ is a closed subset of intS. Since $S \cup \{(r,\theta): r < 1\}$ is a connected, simply connected bounded domain, the Riemann mapping theorem gives a one-to-one conformal map ϕ from $S \cup \{(r,\theta): r < 1\}$ onto the interior U of the unit circle T. Since $g(\partial S) \subset \text{int}S$, $g(\overline{S}) \cup \{(r,\theta): r < 1\} \subset \text{int}S \cup \{(r,\theta): r < 1\}$, hence $\phi(g(\overline{S}) \cup \{(r,\theta): r < 1\})$ is a compact subset of U. So there is a positive number ε so that $\varepsilon < d\left(T,\phi(g(\overline{S}) \cup \{(r,\theta): r < 1\})\right)$. Let $\theta = \phi^{-1}(\{(r,\theta): r = 1 - \varepsilon\})$. Then θ is a simple closed curve in $S \setminus g(S)$ so that $r = 1$ is in its interior and $g(\theta)$ lies in $\text{int}\theta$. $\square$

Before proceeding to the case in which g has exactly one fixed point we will prove the following lemma using the theory of prime ends. The definitions, notations, and theorems used here about prime ends can be found, for example, in [7]. This lemma is also a consequence of the more general Theorem 42 on page 217 of R.L. Moore's <u>Foundations of Point Set Theory</u> [12].

LEMMA 2.5. Suppose each of V and W is a connected, simply connected, bounded domain in E^2 so that $\overline{V}$ and $\overline{W}$ are simply connected and $V \subset W$ and $\partial V \cap \partial W = \{Q\}$, then there is a simple closed curve θ in $(W \setminus \overline{V}) \cup \{Q\}$ so that $\overline{V} \subset \mathrm{int}\theta \cup \{Q\}$.

PROOF. Let $O = W \setminus \overline{V}$. The set O is open and connected, and since Q is in both of $\overline{V}$ and W^c, $\overline{V} \cup W^c$ is connected so that $O = (\overline{V} \cup W^c)^c$ is simply connected. Let ψ be a conformal map of W onto the interior U of the unit circle T (by the Riemann mapping theorem). Every point in $\overline{V}$ except Q is mapped into U. The set $\psi(O)$ is open, connected, and simply connected since O is, and $\psi(V)$ is open, connected, and simply connected since V is. The set $\overline{\psi(V)}$ is a compact, connected subset of $U \cup T$ and $\overline{\psi(V)} \subseteq \psi(\overline{V} \setminus \{Q\}) \cup T$. So, $\overline{\psi(V)} \cap T$ is a closed subset of T; hence $T \setminus \overline{\psi(V)}$ is the union of at most countably many disjoint open arcs each of which is a component of $T \setminus \overline{\psi(V)}$. If $\overline{\psi(V)} \cap T$ is not an arc then there are at least two disjoint open arcs in $T \setminus \overline{\psi(V)}$, say $(\lambda_1, \lambda_1')_{cc}$ and $(\lambda_2, \lambda_2')_{cc}$, so that λ_1, λ_1', λ_2, λ_2' are in $\overline{\psi(V)}$. Let $w_1 \in (\lambda_1, \lambda_1')_{cc}$ and $w_2 \in (\lambda_2, \lambda_2')_{cc}$, let ε be a positive number less than each of $d(w_1, \overline{\psi(V)})$ and $d(w_2, \overline{\psi(V)})$ and finally let z_1 and z_2 be points in $U \cap N(w_1, \varepsilon)$ and $U \cap N(w_2, \varepsilon)$ respectively. Since $N(w_1, \varepsilon) \cap U \subset \psi(O)$ and $N(w_2, \varepsilon) \cap U \subset \psi(O)$ the open segments σ_1 from w_1 to z_1 and σ_2 from w_2 to z_2 are contained in $\psi(O)$. Since z_1 and z_2 are in $\psi(O)$ there is an arc σ_3 from z_1 to z_2 in $\psi(O)$. So $\overline{\sigma_1} \cup \overline{\sigma_2} \cup \overline{\sigma_3}$ does not intersect $\overline{\psi(V)}$ and contains an arc σ from w_1 to w_2. So each of $\sigma \cup [w_1, w_2]_{cc}$ and $\sigma \cup [w_2, w_1]_{cc}$ is a simple closed curve which does not intersect $\psi(\overline{V} \setminus \{Q\})$ hence $\psi(V)$ must lie in the interior of exactly one of these two curves. Suppose $\psi(V)$ lies in the interior of $\sigma \cup [w_1, w_2]_{cc}$, then $\overline{\psi(V)}$ lies in the union of $\sigma \cup [w_1, w_2]_{cc}$

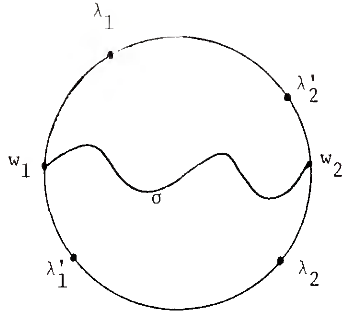

Figure 4. The arc σ in $\overline{U} \setminus \overline{\psi(V)}$.

and its interior but λ_1 and λ_2' lie in the exterior of $L \cup [w_1, w_2]_{cc}$ and are also in $\overline{\psi(V)}$, a contradiction. (Here we are using the Jordan curve theorem.) Thus, $\overline{\psi(V)} \cap T$ is an arc.

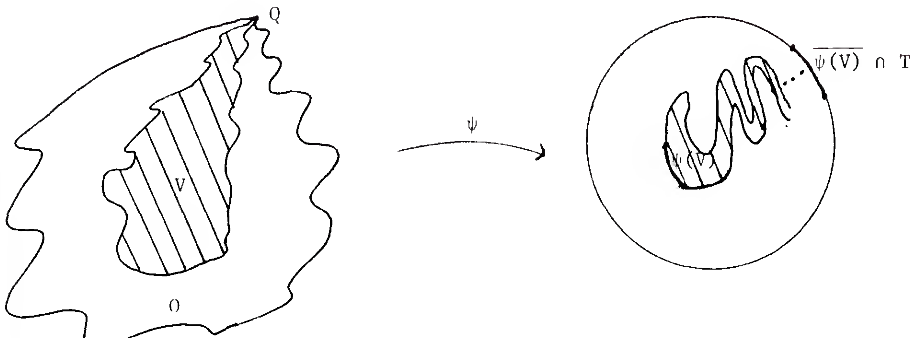

Figure 5. The map ψ.

Now we will show that $\overline{\psi(V)} \cap T$ consists of exactly one point. Suppose $w \in \overline{\psi(V)} \cap T$ and P is a principal point of the prime end P(w) of W which corresponds to w under the map ψ^{-1}. There is a chain $\{\gamma_n\}_{n=1}^{\infty}$ of crosscuts of W belonging to P(w) so that $\{\gamma_n\}_{n=1}^{\infty}$ converges to P. Since

$\{\psi(\gamma_n)\}_{n=1}^{\infty}$ is a chain of crosscuts of U which converges to w, there is a positive integer N so that for $n \geq N$, $\psi(\gamma_n) \cap \psi(V)$ is not empty. So for each $n \geq N$ let x_n be some point of $\psi(\gamma_n) \cap \psi(V)$. Since, for $n \geq N$, $\psi^{-1}(x_n) \in \gamma_n$ and $\{\gamma_n\}_{n=1}^{\infty}$ converges to P, $\{\psi^{-1}(x_n)\}_{n=N}^{\infty}$ is a sequence of points from V which converges to P; hence P must be in $\overline{V}$. Since P must also be in ∂W because the impression of a prime end of W is contained in ∂W, and since $\overline{V} \cup \partial W = \{Q\}$, $P = Q$. That is, the only principal point of any prime end $P(w)$ of W with w in $\overline{\psi(V)} \cap T$ is Q. Now if $C = \psi^{-1}\{r\cdot w: r < 1\}$ then $\overline{C} \setminus C$ is the set of principal points of $P(w)$ so that for each $w \in \overline{\psi(V)} \cap T$, $\lim_{r\to 1^-} \psi^{-1}(r\cdot w)$ exists and is equal to Q. By Fatou's theorem on radial limits [16] for any number α the measure of $\{w \in T: \lim_{r\to 1^-} \psi^{-1}(r\cdot w) = \alpha\}$ is 0; thus, in particular, the measure of $\{w \in \overline{\psi(V)} \cap T: \lim_{r\to 1^-} \psi^{-1}(r\cdot w) = Q\}$ is 0. So $\overline{\psi(V)} \cap T$ is an arc on T whose measure 0; therefore it consists of a single point; call this point w_o. So Q is the sole principal point of the prime end $P(w_o)$ of W.

Next we will use this prime end of W to construct two prime ends of O, each of which will have Q as its only principal point. The points in ∂O which are accessible from O are dense in ∂O, so there are points z_1 and z_2 in $\partial W \setminus \{Q\}$ and $\partial V \cup \setminus \{Q\}$ respectively which are accessible from O. So there are points z_1' and z_2' in O and open arcs L_1 and L_2 in O, whose closures are arcs, from z_1' to z_1 and z_2 to z_2' respectively. Since O is arcwise connected, there is some arc L_3 from z_1' to z_2' in O. Now there is a subset L of $L_1 \cup L_2 \cup L_3$ which is crosscut of O from z_1 to z_2. Let ϕ be a conformal mapping of O onto U. Now $\phi(L)$ is a crosscut of U, so $U \setminus \phi(L)$ has exactly two components; hence $O \setminus L$ has exactly two components, say O_1 and O_2. Let $\{r_i\}_{i=1}^{\infty}$ be a sequence of positive

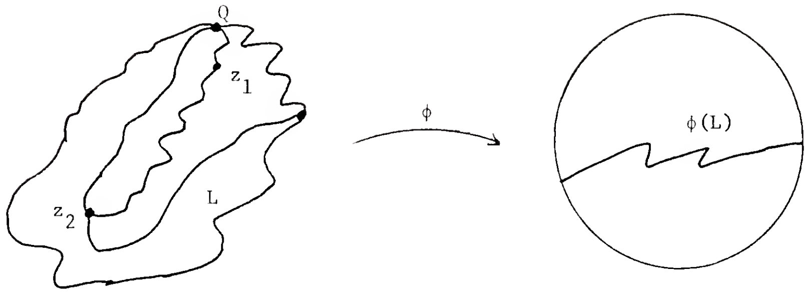

Figure 6. The map ϕ from 0 onto U.

numbers so that the following conditions are met:

(1) $r_i < d(w,\overline{\psi(L)})$,

(2) $r_{i+1} < r_i$,

(3) $\lim_{r\to\infty} r_i = 0$, and

(4) the arclength of $\psi^{-1}(\{z \in U: |z - w| = r_i\})$ approaches 0 as i goes to ∞.

For each positive integer i each of $\{z \in U: |z - w| = r_i\} \cap \psi(0_1)$ and $\{z \in U: |z - w| = r_i\} \cap \psi(0_2)$ is the union of at most countably many disjoint open arcs each of which is a component and each of which has at most one endpoint on T. For each i let α_i be that open arc of $\{z: |z - w| = r_i\} \cap \psi(0_1)$ which has one endpoint on T and let β_i be that open arc of $\{z: |z - w| = r_i\} \cap \psi(0_2)$ which has one endpoint on T. Now each of $\{\psi^{-1}(\alpha_i)\}_{i=1}^{\infty}$ and $\{\psi^{-1}(\beta_i)\}_{i=1}^{\infty}$ is a chain of crosscuts of 0 which belongs to some prime end of 0. In fact each of these chains converges to Q since $\{\psi^{-1}\{z \in U: |z - w| = r_i\}\}_{i=1}^{\infty}$ does (since Q is the only principal point). Also, the chains $\{\psi^{-1}(\alpha_i)\}_{i=1}^{\infty}$ and $\{\psi^{-1}(\beta_i)\}_{i=1}^{\infty}$ belong to different prime ends of 0 since all the crosscuts $\psi^{-1}(\alpha_i)$ belong to

O_1 and all the crosscuts $\psi^{-1}(\beta_i)$ belong to O_2. Let ν and τ be the two points on T so that $P(\nu)$ and $P(\tau)$ are the prime ends of O to which $\{\psi^{-1}(\alpha_i)\}_{i=1}^{\infty}$ and $\{\psi^{-1}(\beta_i)\}_{i=1}^{\infty}$ belong respectively. (The points of T and the prime ends of O correspond via the conformal map ϕ.)

Suppose P is a principal point in the impression of $P(\nu)$, then there is a chain of crosscuts $\{\gamma_n\}_{n=1}^{\infty}$ of O which belongs to $P(\nu)$ and converges to P. If $\{\gamma_n\}_{n=1}^{\infty}$ converges to P, then the set of endpoints of the arcs γ_n has P as its only limit point. Suppose that for all but finitely many positive integers n, γ_n does not have one endpoint in ∂W and the other in ∂V, that is, that there is a positive integer N so that either for all $n \geq N$ both endpoints of γ_n are in ∂W or else for all $n \in N$ both endpoints of γ_n are in ∂V. In the first case, $\{\gamma_n\}_{n=N}^{\infty}$ is a chain of crosscuts of W so belongs to some prime end of W also. Let M be a positive integer so that $M \geq N$ and for all $n \geq M$ the domain D_n corresponding to γ_n does not contain V (considering $\{\gamma_n\}_{n=N}^{\infty}$ as a chain of crosscuts of W). If D_M is the domain corresponding to γ_M considering $\{\gamma_n\}_{n=M}^{\infty}$ as a chain of crosscuts of O, then D_M is also the domain corresponding to γ_M considering $\{\gamma_n\}_{n=M}^{\infty}$ as a chain of crosscuts of W. Thus $\overline{D_M} \subset (O \cup \gamma_M \cup \partial W)$. Since $\{\psi^{-1}(\alpha_i)\}_{i=1}^{\infty}$ and $\{\gamma_n\}_{n=M}^{\infty}$ belong to the same prime end of O, there is a positive integer I so that for $i \geq I$ $\overline{\psi^{-1}(\alpha_i)} \subseteq \overline{D_M}$; but each crosscut $\psi^{-1}(\alpha_i)$ has one endpoint in $\partial V \setminus Q$, a contradiction since $\overline{D_M} \cap (\partial V \setminus Q) = \phi$. In the second case an analogous argument using the fact that $\{\gamma_n\}_{n=N}^{\infty}$ also belongs to a prime end of $\overline{V}^c$ leads to a contradiction. Hence there is an infinite sequence J of positive integers so that $\{\gamma_n\}_{n \in J}$ is a chain of crosscuts each of which has one endpoint on ∂V and the other on ∂W. Since $\{\gamma_n\}_{n \in J}$ converges to P, $P \in \partial V \cap \partial W = \{Q\}$ so that $P = Q$. That is, Q is the only

principal point in the prime end of O determined by $\{\psi^{-1}(\alpha_i)\}_{i=1}^{\infty}$. By the same argument, Q is the only principal point in the prime end of O determined by $\{\psi^{-1}(\beta_i)\}_{i=1}^{\infty}$.

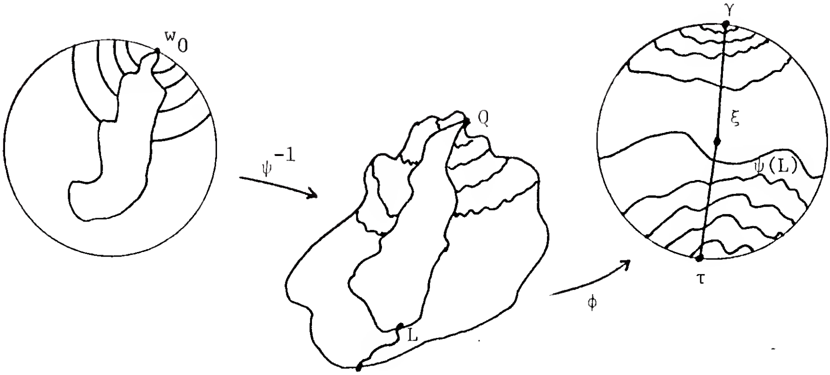

Figure 7. Chains belonging to the two prime ends.

Now let ξ be the union of the open line segment ξ_1 from the origin to ν, the open line segment ξ_2 from τ to the origin, and the origin. Since Q is the only principal point of each $P(\nu)$ and $P(\tau)$, $\phi^{-1}(\xi_1)$ is an endcut of O to Q and $\phi^{-1}(\xi_2)$ is an endcut of O to Q. So $\phi^{-1}(\xi)$ is a crosscut of O and $\phi^{-1}(\xi) \cup \{Q\}$ is a simple closed curve in $O \cup \{Q\}$. Let $\theta = \phi^{-1}(\xi) \cup \{Q\}$. It remains to show that $\bar{V} \subset (\mathrm{int}\theta \cup \{Q\})$. Let $\theta\colon [0,1] \to E^2$ be a parametrization of θ which is one-to-one except that $\theta(0) = \theta(1) = Q$ and so that $\theta([0,\tfrac{1}{2}]) = \overline{\phi^{-1}(\xi_1)}$ and $\theta([\tfrac{1}{2},1]) = \overline{\phi^{-1}(\xi_2)}$. There is a number r and a sequence of numbers $\{t_i\}_{i=1}^{\infty}$ so that the following hold:

(1) $0 < t_{i+1} < t_i < r$,

(2) $\theta(t_i) \in \psi^{-1}(\alpha_i)$, and

(3) $\theta((0,t_i)) \subset D_i$ where D_i is the domain corresponding to $\psi^{-1}(\alpha_i)$.

And there is a sequence of numbers $\{s_i\}_{i=1}^{\infty}$ so that the following hold:

(1) $r < s_i < s_{i+1} < 1$,

(2) $\theta(s_i) \in \psi^{-1}(\beta_i)$, and

(3) $\theta((s_i,1)) \subset E_i$ where E_i is the domain corresponding to $\psi^{-1}(\beta_i)$.

These sequences exist since $\theta([0,r])$ is an endcut of O which converges to the prime end $P(\nu)$ and $\theta([r,1])$ is an endcut of O which converges to the prime end $P(\tau)$. Let λ_i be that subarc of $T \setminus (\overline{\alpha}_i \cup \overline{\beta}_i)$ which contains w_o. Since $\psi(\theta[t_i,s_i])$ is a simple arc in U, $\psi(\theta[t_i,s_i]) \cap \overline{\alpha}_i$ contains $\psi(\theta(t_i))$ and $\psi(\theta[t_i,s_i]) \cap \overline{\beta}_i$ contains $\psi(\theta(s_i))$, $C_i = \psi(\theta[t_i,s_i]) \cup \alpha_i \cup \beta_i \cup \overline{\lambda}_i$ is a closed curve in $U \cup T$. There is a simple arc $\hat{C}_i$ from $\overline{\alpha}_i \cap T$ to $\overline{\beta}_i \cap T$ in C_i, so that $\hat{C}_i \cup \lambda_i$ is a simple closed curve in $U \cup T$; let $\beta_i = (\hat{C}_i \cup \lambda_i) \cup \operatorname{int}(\hat{C}_i \cup \lambda_i)$. Now, $\overline{\psi(V)} \setminus \{w_o\}$ does not intersect $\hat{C}_i \cup \lambda_i$ and $w_o \in \lambda_i$ is a limit point of $\psi(V)$; hence $\overline{\psi(V)}\setminus\{w_0\}$ must lie in $\operatorname{int}\beta_i$ (using the Jordan curve theorem). Since $\psi((\theta \cup \operatorname{int}\theta) \setminus \{Q\}) \cup \{w_0\} = \bigcap_{i=1}^{\infty} \beta_i$, $\overline{\psi(V)} \setminus \{w_o\} \subset \psi(\theta \cup \operatorname{int}\theta \setminus \{Q\}) \cup \{w_o\}$ so that $\overline{V} \subset \theta \cup \operatorname{int}\theta$ and since $\theta \subset O \cup \{Q\}$, $\overline{V} \subset \operatorname{int}\theta \cup \{Q\}$. □

LEMMA 2.6. Suppose g has exactly one fixed point F in A. If no δ-chain terminates, then there is a simple closed curve θ in $\operatorname{int}A$ with $r = 1$ in its interior and $g(\theta) \subset \operatorname{int}\theta \cup F$ (so that the annulus bounded by $r = 1$ and θ is mapped onto a proper subset of itself).

PROOF. Construct the set S as in lemma 2.3. Let P be a point in ∂S which is not F; in this case $\delta(g(P)) > 0$. There is a positive number $\varepsilon_1 < \frac{\delta(g(P))}{4}$ so that if $d(g(P),g(Q)) < \varepsilon_1$ then $\delta(g(Q)) \geq \frac{3}{4}\,\delta(g(P))$.

There is a positive number ε_2 so that if $d(P,Q) < \varepsilon_2$ then $d(g(P),g(Q) < \varepsilon_1$. Since P is a limit point of M, there is a point Q in M so that $d(P,Q) < \varepsilon_2$. Thus for this Q, $d(g(Q)) \geq \frac{3}{4}\,\delta(g(P))$. Also, there is a positive integer n so that $Q \in M_n$, hence $g(Q) \in M_{n+1}$. Since no δ-chain terminates $N\big(g(Q),\ \delta(g(Q))\big) \cap \{(r,\theta)\colon r > 1\} \subset M_{n+1} \subset A$. Since $d(g(P),g(Q)) < \frac{\delta(g(P))}{4}$ and $\delta(g(Q)) \geq \frac{3}{4}\,\delta(g(P))$, $N\big(g(P),\ \frac{\delta(g(P))}{4}\big) \subset N\big(g(Q),\ \delta(g(Q))\big)$, hence $g(P) \in \operatorname{int}S$. Therefore, $g(\partial S) \subset S \cup \{F\}$.

Now consider the sets $W = \{(r,\theta)\colon 0 \leq r < 1\} \cup S$ and $V = \{(r,\theta)\colon 0 \leq r < 1\} \cup g(S)$. If ∂S contains the point F, then the hypotheses of lemma 2.5 are satisfied where the point Q is F. So there is a simple closed curve θ in $(W \setminus \overline{V}) \cup \{F\}$ which has $\overline{V} \setminus \{F\}$ in its interior. Therefore $\theta \subset (S \setminus \overline{g(S)}) \cup \{F\}$, and $g(\theta) \subset (g(S) \cup \{F\})$ so that $g(\theta) \subset \operatorname{int}\theta \cup \{F\} \subset \operatorname{int}\theta \cup \theta$. If $F \notin \partial S$, then a simple closed curve θ with the required properties can be constructed by the method used in lemma 2.4. $\square$

Putting lemmas 2.4 and 2.6 together we immediately have the following proposition.

PROPOSITION 2.7. Let g be a twist homeomorphism of the annulus A onto g(A) with at most one fixed point. If there is no simple closed curve θ in intA with $r = 1$ in its interior and with $g(\theta) \subset \theta \cup \operatorname{int}\theta$, then there is a terminating δ-chain.

Since it was assumed in the first section that g has at most one fixed point and that no such simple closed curve θ exists, there must exist some terminating δ-chain. Hence there is such a δ-chain of minimal length, so let $P_0, P_1, \ldots, P_{N-1}, P_N'$ be a terminating δ-chain of minimal length. If $P_N' = g(P_{N-1})$, then let $P_N = P_N'$. Otherwise, let

$\varepsilon = \sup\{\alpha\colon N(g(P_{N-1}),\alpha) \cap \gamma = \phi\}$, so that $\varepsilon < \delta(g(P_{N-1}))$, and $\overline{N(g(P_{N-1}),\varepsilon)} \cap \gamma \neq \phi$ since $N(g(P_{N-1}), \delta(g(P_{N-1}))) \cap \gamma \neq \phi$. Let $P_N \in \overline{N(g(P_{N-1}),\varepsilon)} \cap \gamma$, then $P_0, P_1, \ldots, P_{N-1}, P_N$ is also a terminating δ-chain of minimal length. For each i, $0 \le i \le N-1$, there is some point V_i so that $P_{i+1} = g(P_i) + V_i$ and $\|V_i\| < \delta(g(P_i))$; let $\delta_i = \|V_i\|$. By minimality, the points $P_0, P_1, \ldots, P_{N-1}$ and P_N are distinct, so the points $g(P_0), g(P_1), \ldots, g(P_{N-2})$ and $g(P_{N-1})$ must also be distinct and $g(P_i)$ may be P_j only if $j = i + 1$. Further, if $i > j$, then P_{i+1} is not in $N(g(P_j),\delta(g(P_j)))$ since then $P_0, \ldots, P_j, P_{i+1}, \ldots, P_N$ would be a terminating δ-chain, contradicting the minimality of the original. If $i < N-1$ then $N(g(P_i),\delta(g(P_i)))$ does not intersect γ, since if it did, $P_0, P_1, \ldots P_i, P'_{i+1}$ would be a terminating δ-chain for any point P'_{i+1} in $N(g(P_i),\delta(g(P_i))) \cap \gamma$, again contradicting the minimality of $P_0, P_1, \ldots, P_N$. So if $i < N-1$, since $\overline{N(g(P_i),\delta_i)} \subset N(g(P_i),\delta(g(P_i)))$, $\overline{N(g(P_i),\delta_i)}$ does not intersect γ. If $i = N-1$, then we have $N(g(P_i),\delta_i)$ does not intersect γ (unless $\delta_i = 0$) but $\overline{N(g(P_i)\delta_i)}$ does intersect γ. In the next section $P_0, P_1, \ldots, P_N$ will denote a terminating δ-chain with these minimality properties.

If g is a twist homeomorphism of the annulus A onto g(A) with at most one fixed point, then g^{-1} is a twist homeomorphism of g(A) onto A. For g^{-1} an appropriate function δ can be chosen and the same sequence of arguments shows that if there is no simple closed curve θ in int(g(A)) with $r = 1$ in its interior and $g^{-1}(\theta) \subset \operatorname{int}\theta \cup \theta$, then there is a terminating δ-chain for g^{-1}. And so a terminating δ-chain for g^{-1} can be found with the minimality properties described above.

The Auxiliary Homeomorphism T

Using the minimality properties of the δ-chain $P_0, P_1, \ldots, P_N$ discussed at the end of the last section, a homeomorphism T will now be constructed so that $T(g(P_i)) = P_{i+1}$ for $0 \leq i \leq N-1$, so that $T \cdot g$ has exactly the same fixed points as g and so that $T \cdot g$ is homotopic to g. A sequence of technical lemmas is necessary for the construction of T. Then, since we want to do all the index computations in the Cartesian plane, we lift the homotopy between $T \cdot g$ and g to a homotopy between a certain lift of $T \cdot g$ and h.

LEMMA 2.8. There is a collection of polygonal arcs α_i, for $1 \leq i \leq N$, with the following properties:

(1) $\alpha_i(0) = g(P_{i-1})$ and $\alpha_i(1) = P_i$,

(2) $\alpha_i \subset \mathrm{int}A \cup \{g(P_0), P_N\}$,

(3) $\alpha_i \subset N(g(P_{i-1}), \delta_{i-1}) \cup \{P_i\}$, and

(4) $\alpha_i \cap \alpha_j = \phi$ if $i \neq j$.

PROOF. For $1 \leq k \leq N$, the collection of arcs $A_k = \{\alpha(i,k): 1 \leq i \leq k\}$ will be called a <u>set of connecting arcs</u> for the subchain $P_0, P_1, \ldots P_k$ of $P_0, P_1, \ldots, P_N$ if the arcs $\alpha(i,k)$ satisfy the following conditions:

(i) $\alpha(i,k)$ is a simple polygonal arc from $g(P_{i-1})$ to P_i,

(ii) $\alpha(i,k) \subset \mathrm{int}A \cup \{g(P_0), P_N\}$,

(iii) $\alpha(i,k) \subset N(g(P_{i-1}), \delta_{i-1}) \cup \{P_i\}$,

(iv) $\alpha(i,k) \cap \alpha(j,k) = \phi$ if $i \neq j$, and

(v) $\alpha(i,k)$ does not contain any $g(P_\ell)$ or $P_{\ell+1}$ with $k \leq \ell \leq N$.

The proof proceeds by induction. By minimality P_1 must be in $r > 1$. Let $\alpha(1,1)$ be a polygonal arc from $g(P_0)$ to P_1 in $(\mathrm{int}A \cup \{g(P_0)\}) \cap \big(N(g(P_0), \delta_0) \cup \{P_1\}\big)$ which misses every point $g(P_\ell)$ or $P_{\ell+1}$ with $1 \leq \ell \leq N$. Let $A_1 = \{\alpha(1,1)\}$, then A_1 is a set of connecting

arcs for the subchain P_0, P_1. Suppose a set A_k of connecting arcs for the subchain $P_0, P_1, \ldots, P_k$ with $k < N$ has been constructed. Now we will construct a set A_{k+1} of connecting arcs for $P_0, P_1, \ldots, P_{k+1}$. If $P_{k+1} = g(P_k)$, then for each $j \leq k$ let $\alpha(j,k+1) = \alpha(j,k)$ and let $\alpha(k+1,k+1)\colon [0,1] \to E^2$ be the constant map $\alpha(k+1,k+1)(t) = P_{k+1}$. Now $A_{k+1} = \{\alpha(j,k+1)\colon 1 \leq j \leq k+1\}$ is a set of connecting arcs for P_0, P_1, $\ldots, P_{k+1}$ and we are done. So suppose $g(P_k) \neq P_{k+1}$. Now let $D = N(g(P_k),\delta_k) \cap A$. Since $D \subset N(g(P_k),\delta(g(P_k))$ and $N\big(g(P_k),\ \delta(g(P_k))\big) \cap \gamma = \phi$ since the δ-chain is minimal, D does not intersect γ as long as $k < N$, and if $k+1 < N$ then $\overline{D}$ does not intersect γ. Note that D may be the interior of a disk or D may be the intersection of the interior of a disk with $r \geq 1$. Since each $\alpha(i,k)$, $1 \leq i \leq k$, is polygonal and since the arcs $\alpha(i,k)$, $1 \leq i \leq k$, are disjoint, $\left(\bigcup_{i=1}^{k} \alpha(i,k)\right) \cap D$ is the union of at most finitely many disjoint polygonal arcs β so that $\overline{\beta} \setminus \beta \subset \overline{D} \setminus D$, where each arc β is a component and may be open, half-open or closed. Let B be the collection of all such arcs. Note that if $\beta \in B$, then $\overline{\beta}$ does not contain P_{k+1} or $g(P_k)$. Let B_1 be the collection of all arcs in B which do not separate P_{k+1} from $g(P_k)$ in $D \cup \{P_{k+1}\}$ and let B_2 be the collection of all arcs in B which do separate P_{k+1} from $g(P_k)$. Now since there are only finitely many arcs in $B_1 \cup B_2$, and none of them contain either of $g(P_k)$ or P_{k+1}, there is a simple polygonal arc $\alpha_{k+1}\colon [0,1] \to \mathrm{int}D \cup \{P_k\}$ so that $\alpha_{k+1}(0) = g(P_k)$, $\alpha_{k+1}(1) = P_{k+1}$, α_{k+1} does not intersect any arc in B_1 or contain any of the points $g(P_\ell)$ or $P_{\ell+1}$ for $k+1 \leq \ell \leq N$, and, moreover, for each $\beta \in B_2$, α_{k+1} intersects β in exactly one point. For each arc β in B_2 there is exactly one integer $j \leq k$ so that β is a subarc of $\alpha(j,k)$. Next we are going to replace each arc β in B_2 by a

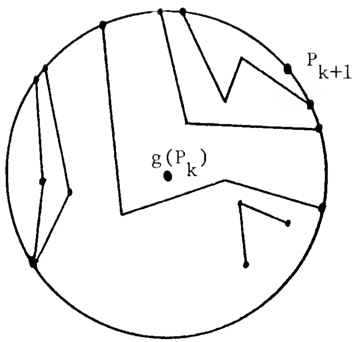

Figure 8. The set D.

new arc β' which does not intersect α_{k+1}, and which is contained in $N(g(P_{j-1}),\delta_{j-1}) \cup \{P_j\}$.

First, if $\beta \in B_2$ then the component of $D \setminus \overline{\beta}$ which contains $g(P_k)$ is a subset of $N(g(P_{j-1}),\delta_{j-1})$ if β is a subarc of $\alpha(j,k)$. The argument for this which follows uses the minimality of the δ-chain $P_0, P_1, \ldots, P_k$ and the geometry of the sets $N(g(P_i),\delta_i) \cap A$. Since β separates $g(P_k)$ from P_{k+1}, $\overline{\beta}$ has exactly two endpoints on the boundary of D, both of which are in $r > 1$ if $j \neq 1$ and at least one of which is in $r > 1$ if $j = 1$. Since $\overline{\beta} \subset \overline{N(g(P_{j-1}),\delta_{j-1})} \cap A$, $\overline{N(g(P_{j-1}),\delta_{j-1})} \cap A$ intersects the part of the boundary of D which lies in $r > 1$ in at least two points if $j > 1$ and in at least one point if $j = 1$. Since $P_{k+1} \notin \overline{N(g(P_{j-1}),\delta_{j-1})}$ by minimality, and $P_{k+1} \in \overline{D}$, $\overline{D} \not\subseteq \overline{N(g(P_{j-1}),\delta_{j-1})} \cap A$. If $\overline{N(g(P_{j-1}),\delta_{j-1})} \cap A$ is a subset of $\overline{D}$, then there are three possible types of configurations since $N(g(P_i),\delta_i)$ lies in $\text{int}\gamma$ for all $i \le N$: Either: (1) Both D and $N(g(P_{j-1}),\delta_{j-1}) \cap A$ are disks, that is, $D \cap \{(r,\theta): r = 1\} = \phi$ and $N(g(P_{j-1}),\delta_{j-1}) \cap \{(r,\theta): r = 1\} = \phi$; (2) both D and $N(g(P_{j-1}),\delta_{j-1}) \cap A$

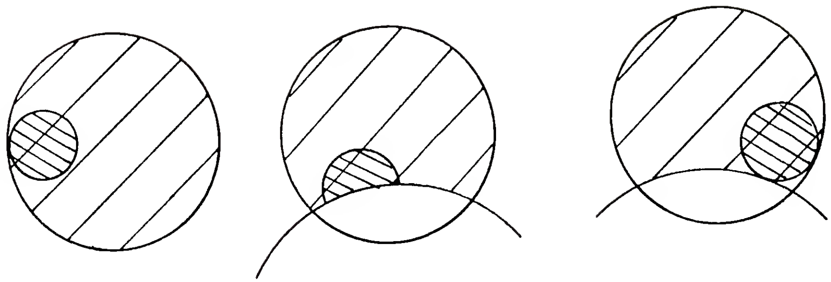

Figure 9. The three possible types of configurations.

are "truncated disks," that is, $D \cap \{(r,\theta): r = 1\} \neq \phi$ and $N(g(P_{j-1}),\delta_{j-1}) \cap \{(r,\theta): r = 1\} \neq \phi$; or (3) $\overline{N(g(P_{j-1}),\delta_{j-1})} \cap A$ is a disk and D is a "truncated disk." If one disk lies in another (case 1), then the boundary of the first intersects the boundary of the second in one, none, or all of its points. If the configuration of D and $\overline{N(g(P_{j-1}),\delta_{j-1})} \cap A$ is of this type, then $j > 1$ so that $\overline{N(g(P_{j-1}),\delta_{j-1})} \cap A$ has at least two points in $r > 1$ in common with the boundary of D, but $\overline{N(g(P_{j-1}),\delta_{j-1})} \cap A$ does not contain all of the boundary of D, a contradiction. If a "truncated disk" lies in a "truncated disk" (case 2), then the boundary of the first intersects the boundary of the second in one, none, or all points of the boundary of the second which lie in $r > 1$. If $\overline{N(g(P_{j-1})\delta_{j-1})} \cap A$ intersects the boundary of D in exactly one point in $r > 1$, then $j = 1$ and β is a subarc of $\alpha(1,k)$ which contains $g(P_0)$. But $N(g(P_0), \delta_0) \cap A$ is a "truncated disk" with its center on $r = 1$, while D is a "truncated disk" with its center in $r > 1$, so that it is not possible for the boundaries of the two to have exactly one point in common in

$r > 1$. So $\overline{N(g(P_j),\delta_j)} \cap A$ must intersect the boundary of D in $r > 1$ in none or all of its points, again a contradiction. If a disk lies in a truncated disk (case 3), the first intersects that part of the boundary of the second which lies in $r > 1$ in one or no points. Since $j > 1$, $\overline{N(g(P_{j-1}),\delta_{j-1})} \cap A$ contains at least two points which lie in the boundary of D in $r > 1$, but does not contain all of the boundary of D in $r > 1$; this configuration is also impossible. Hence $\overline{N(g(P_{j-1}),\delta_{j-1})} \cap A$ is not a subset of $\overline{D}$. Thus, so far we have shown that $\overline{D} \not\subseteq \overline{N(g(P_{j-1}),\delta_{j-1})} \cap A$ and $\overline{N(g(P_{j-1}),\delta_{j-1})} \cap A \not\subseteq \overline{D}$ for any j so that $\alpha(j,k)$ contains a subarc which belongs to B_2. So the boundary of $\overline{N(g(P_{j-1}),\delta_{j-1})} \cap A$ intersects intD in an open circular arc C_j which does not intersect β, although $\overline{\beta}$ may contain P_{j+1}, in which case P_{j+1} must be on the boundary of D since β separates D, and in this case $\overline{C_j}$ would intersect $\overline{\beta}$. So intD $\setminus C_j$ has two components each of which is bounded by a simple closed curve consisting of the union of either two or three circular arcs. One of these two components, namely $N(g(P_{j-1}),\delta_{j-1}) \cap$ intD, contains β (except for the point $g(P_0)$ if $j = 1$). If $g(P_k)$ were in the closure of the other component of intD $\setminus C_j$, then it would not be separated from P_{k+1} by β so $g(P_k)$ must be in $N(g(P_{j-1}),\delta_{j-1}) \cap$ intD. So the component of D $\setminus \beta$ which contains $g(P_k)$ is a subset of $N(g(P_{j-1}),\delta_{j-1})$.

Now let $J = \{j \le k$: there is some $\beta \in B_2$ which is a subarc of $\alpha(j,k)\}$. There is a positive number ε so that:

(1) $N(g(P_k),2\varepsilon) \subset N(g(P_{j-1}),\delta_{j-1})$ for all $j \in J$,

(2) $\varepsilon < \frac{1}{2}\min\{d(\alpha_{k+1},\beta)\colon \beta \in B_1\}$, and

(3) $\varepsilon < \frac{1}{2}\min\{d(\alpha_{k+1},g(P_\ell))\colon k+1 \le \ell \le N\}$.

(Note that by minimality $P_\ell \notin \overline{D}$ if $k+1 \le \ell \le N$.) Since each of the arcs β in B_2 separates P_{k+1} from $g(P_k)$ and each intersects the arc α_{k+1} from

$g(P_k)$ to P_{k+1} only once there is a sequence of numbers $0 < t_1 < t_2 < \cdots < t_p < 1$ so that for each i, $1 \le i \le p$, $\alpha_{k+1}(t_i) \in \beta$ for exactly one β in B_2. Label the arcs in B_2 by requiring $\alpha_{k+1}(t_i) \in \beta_i$. Note that for each i, $1 < i < p$, β_i separates β_{i-1} from β_{i+1}. There is a polygonal simple closed curve ψ which lies in $N(\alpha_{k+1}([0,t_p]),\varepsilon) \cap \mathrm{int}D$ which contains $\alpha_{k+1}([0,t_p])$ in its interior and which intersects each arc β in B_2 exactly twice. Now let r_i and s_i be the two points in $\beta_i \cap \psi$, $1 \le i \le p$. The component O_1 of $\mathrm{int}D \setminus (\beta_1 \cup \psi)$ which contains $g(P_i)$ is a subset of $N(g(P_{j-1}),\delta_{j-1})$ for that $j \in J$ which has β_1 a subset of $\alpha(j,k)$. There is a simple polygonal arc τ_1 from r_1 to s_1 in $O_1 \setminus \alpha_{k+1}$. Let β_1' be the arc which is obtained by replacing the subarc of β_1 from r_1 to s_1 by τ_1. Let ψ_1 be the union of τ_1 with that part of ψ which is not in the boundary of O_1. Note that β_1' lies in $N(g(P_{j-1}),\delta_{j-1})$, and does not intersect $\bigcup_{i\neq j}\alpha(i,k)$ nor any of the points $g(P_\ell)$, $P_{\ell+1}$ for $1+k \le \ell \le N$, by choice of ε. Now, the component O_2 of $\mathrm{int}D \setminus (\beta_2 \cup \psi_1)$ which contains $g(P_k)$ lies in $N(g(P_{j-1})\delta_{j-1})$ for that $j \in J$ which has β_2 a subarc of $\alpha(j,k)$. There is a simple polygonal arc τ_2 from r_2 to s_2 in $O_2 \setminus \alpha_{k+1}$. Let β_2' be the arc which is obtained from β_2 by replacing the subarc from r_2 to s_2 by β_2', and let ψ_2 be the union of τ_2 with that part of ψ_1 which is not part of the boundary of O_2. Proceeding by induction we produce new polygonal arcs $\beta_1', \ldots \beta_p'$ which do not intersect α_{k+1}. Let $\alpha(j,k+1)$ be the polygonal arc obtained from $\alpha(j,k)$ by replacing every $\beta_i \in B_2$ which is a subarc of $\alpha(j,k)$ (if any) by β_i'. The resulting simple polygonal arcs $\alpha(j,k+1)$, $1 \le j \le k$ have all the properties that the arcs $\alpha(j,k)$ had and in addition have the property that $\alpha_{k+1} \cap \alpha(j,k+1) = \phi$ for $1 \le j \le k$. So let $\alpha(k+1,k+1) = \alpha_{k+1}$, then $A_{k+1} = \{\alpha(j,k+1)\colon 1 \le j \le k+1\}$ is a set of connecting arcs for the

subchain $P_0, P_1, \ldots P_{k+1}$. Hence there is a collection A_N of connecting arcs for $P_0, P_1, \ldots P_N$; letting $\alpha_i = \alpha(i,N)$ for $1 \le i \le N$ produces the required polygonal arcs. □

LEMMA 2.9. If α is a simple closed curve in E^2 and if p and q are two points in intα, then there is a homeomorphism $T\colon \overline{\text{int}\alpha} \twoheadrightarrow \overline{\text{int}\alpha}$ and a homotopy $I\colon [0,1] \times \overline{\text{int}\alpha} \twoheadrightarrow \overline{\text{int}\alpha}$ which have the following properties:

(1) $T(p) = q$ and $T(w) = w$ for all w in α,

(2) $I(1,z) = T(z)$ for all z in $\overline{\text{int}\alpha}$ and $I(0,z) = z$ for all z in $\overline{\text{int}\alpha}$,

(3) $I(t,w) = w$ for all (t,w) in $[0,1] \times \alpha$, and

(4) I_t is a homeomorphism of $\overline{\text{int}\alpha}$ onto $\overline{\text{int}\alpha}$ for each $t \in [0,1]$.

PROOF. There is a homeomorphism ϕ from $\overline{\text{int}\alpha}$ onto the rectangle in E^2 with vertices $(-1,-1)$, $(-1,1)$, $(1,1)$, and $(1,-1)$, which maps p onto $(0,-\frac{1}{2})$ and q onto $(0,\frac{1}{2})$. Now let $\tau_{-1}(y) = y$, $\tau_1(y) = y$, and

$$\tau_0(y) = \begin{cases} 3y + 2 & \text{if } -1 \le y \le -\frac{1}{2} \\ \frac{1}{3}y + \frac{2}{3} & \text{if } -\frac{1}{2} < y \le 1. \end{cases}$$

If $-1 < x < 0$, then there is a λ with $0 < \lambda < 1$ so that $x = \lambda\cdot 0 + (1-\lambda)(-1)$; so let $\tau_x(y) = \lambda\tau_0(y) + (1-\lambda)\tau_{-1}(y)$. If $0 < x < 1$, then there is a λ with $0 < \lambda < 1$ so that $x = \lambda\cdot 1 + (1-\lambda)\cdot 0$; so let $\tau_x(y) = \lambda\tau_1(y) + (1-\lambda)\,\tau_0(y)$. Now let $\tau(x,y) = (x,\tau_x(y))$; then τ is a homeomorphism of the rectangle onto itself, and $\Phi(t,(x,y)) = \Phi_t(x,y) = (x,y) + t(0,\tau_x(y)-y)$ is a homotopy with $\Phi(0,(x,y)) = (x,y)$, and $\Phi(1,(x,y)) = \tau(x,y)$. So let $T = \phi^{-1}\tau\phi$ and let $I_t = \phi^{-1}\Phi_t\phi$ for each $t \in [0,1]$; then T and I have the required properties. □

LEMMA 2.10. There is a collection of disjoint closed (topological) disks, D_j, $1 \leq j \leq N$, so that for the arcs α_j constructed in lemma 2.8 the following hold:

(1) $\alpha_1 \setminus \{g(P_0)\} \subset \mathrm{int}D_1 \subseteq \overline{D_1} \subseteq N\big(g(P_0),\delta(g(P_0))\big) \cap A$ where the disk D_1 has exactly one subarc of $r = 1$ as part of its boundary and $g(P_0)$ is not an endpoint of that subarc,

(2) $\alpha_j \subset \mathrm{int}D_j \subset \overline{D_j} \subset N\big(g(P_{j-1}),\delta(g(P_{j-1}))\big) \cap \mathrm{int}A$ for $1 < j < N$, and

(3) $\alpha_N \subset \mathrm{int}D_N \subset \overline{D_N} \subset N\big(g(P_{N-1}),\delta(g(P_{N-1}))\big)$.

PROOF. Let ε be a positive number less than each of $\frac{1}{2}\min\{d(\alpha_i,\alpha_j): i \neq j\}$, $\frac{1}{2}\min\{d(\alpha_i,\{(r,\theta): r = 1\}): 1 < i \leq N\}$, and $\frac{1}{2}\min\{d(\alpha_i,\gamma): 1 \leq i < N\}$. For $1 < j \leq N$ there is a closed topological disk D_j so that $\alpha_j \subset \mathrm{int}D_j \subset D_j \subset N(\alpha_j,\varepsilon) \cap N(g(P_{j-1}),\delta(g(P_{j-1})))$. Points z_1 and z_2 on $r = 1$ can be found so that the interval $[z_1,g(P_0),z_2]c$ of $r = 1$ is contained in $N(\alpha_1,\varepsilon) \cap N(g(P_0),\ \delta g(P_0))$. There is a simple polygonal arc from z_2 to z_1 in $N(\alpha_1,\varepsilon) \cap N(g(P_0),\delta(g(P_0))) \cap \mathrm{int}A$ which does not intersect α_1. If β is the union of these two arcs, then $\alpha_1 \setminus \{g(P_0)\} \subset \mathrm{int}\beta$ so that $D_1 = \mathrm{int}\beta \cup \beta = \overline{\mathrm{int}\beta}$ is a disk which has the properties required in (1). The disks D_j, $1 \leq j \leq N$, are disjoint by the first condition on ε. $\square$

LEMMA 2.11. There is a positive number η so that the following hold:

(1) the half-open annulus $1 < r \leq 1 + \eta$ intersects exactly one segment of the polygonal arc α_1 and exactly two segments of the polygonal curve $\partial D_1 \cap \mathrm{int}A$,

(2) $\eta < \frac{1}{2}\min\big\{d(\{(r,\theta): r = 1\},D_j): 2 \leq j \leq N\big\}$,

(3) $\eta < \frac{1}{2}\, d(g(P_0),\partial D_1 \cap \mathrm{int}A)$,

(4) $\eta < \frac{1}{2}\min\{\delta(P): P \text{ is on } r = 1\}$,

(5) $d(P,g^{-1}(P)) > 2\eta$ for all P in the annulus $1 \le r \le 1 + \eta$, and

(6) $d(P,h(P)) > 3\eta$ for all P in $y \le \eta$.

PROOF. Let η_1 be a positive number so that $1 < r \le 1 + \eta_1$ intersects only one segment of α_1 and exactly two segments of $\partial D_1 \cap \mathrm{int}A$. Since δ is continuous and positive on the compact set $r = 1$, there is a number ε so that $\delta(P) > \varepsilon$ for all P in $r = 1$. Since δ is continuous on A, there is a positive number η^* less than each of $\varepsilon/2$, η_1, $\frac{1}{2}\min\{d(\{(r,\theta): r = 1\}, D_j): 2 \le j \le N\}$, and $\frac{1}{2}d(g(P_0), \partial D_1 \cap \mathrm{int}A)$ so that $\delta(P) > \varepsilon$ for all P in the annulus $1 \le r \le 1 + \eta^*$. Since h is periodic and fixed point free on $y=0$, there is a positive number ε' so that $d(P,h(P)) > \varepsilon'$ for all P in $y=0$ and there is a positive number η' so that if P is in $y \le \eta'$, $d(P,h(P)) \ge 3\eta'$. Now choose $\eta < \min(\eta',\eta^*)$. Hence, if P is in $1 \le r \le 1 + \eta$, then $\delta(P) > \varepsilon > 2\eta$. Since $\overline{N(P,\delta(P))} \cap g^{-1}\,\overline{N(P,\delta(P))} = \phi$, $\overline{N(P,2\eta)} \cap g^{-1}\,\overline{N(P,2\eta)} = \phi$ for P in $1 \le r \le 1 + \eta$, so that $d(P,g^{-1}(P)) > 2\eta$. $\square$

Next, consider $D_1 \cup \{(r,\theta): 1 - \eta \le r \le 1 + \eta\}$. This is a compact subset of E^2 which does not intersect D_j for $1 < j \le N$. Let z_1 and z_2 be the points on $r = 1$ so that $[z_1,g(P_0),z_2]c$ is $\partial D_1 \cap \{(r,\theta): r = 1\}$. There are points v_1 and v_2 on $r = 1 + \eta$ so that $\{v_1,v_2\} = \partial D_1 \cap \{(r,\theta): r = 1 + \eta\}$ and v_1 is on the same segment of ∂D_1 that z_1 is and v_2 is on the same segment of ∂D_1 as z_2 is. Now let w_1 and w_2 be the points which are on the intersection of $r = 1$ with the radials through v_1 and v_2, respectively, and let u_1, u_2 be the points in the intersection of $r = 1 - \eta$ with the radials through v_1 and v_2, respectively. The segment $[w_1,g(P_0),w_2]_c$ occurs on $r = 1$ since $\eta < \frac{1}{2}d(g(P_0),\partial D_1 \cap \mathrm{int}A)$ by part (3) of lemma 2.11. Let β be the closed

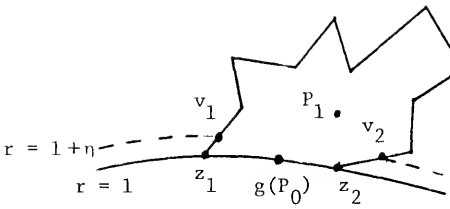

Figure 10. The set D_1.

curve which is the union of the radial segments from u_1 to v_1 and from u_2 to v_2, the interval $[u_1,u_2]c$ on $r = 1 - \eta$ and $\partial D_1 \cap \{(r,\theta)\colon r \geq 1 + \eta\}$. The curve β is simple since η satisfies part (1) of lemma 2.11 and α_1 is a subset of intβ. Finally, let $E_1 = \text{int}\beta \cup \beta$, and let $E_0 = \overline{\{(r,\theta)\colon 1 - \eta < r < 1 + \eta\} \setminus \text{int}E_1}$. In the next two lemmas homeomorphisms T_0 and T_1 and homotopies I_0 and I_1 will be constructed for E_0 and E_1.

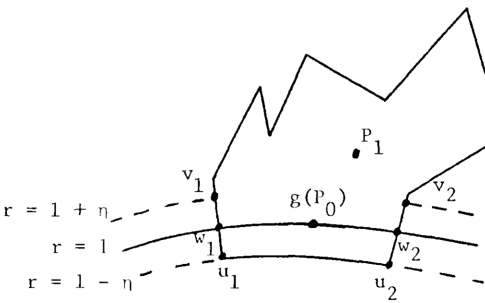

Figure 11. The set E_1.

LEMMA 2.12. There is a homeomorphism T_0: $E_0 \twoheadrightarrow E_0 \cap \{(r,\theta): 1 \le r \le 1 + \eta\}$ and a homotopy I_0: $[0,1] \times E_0 \to E_0$ which have the following properties:

(1) $T_0(P) = P$ if P is in $r = 1 + \eta$, and $d(P,T_0(P)) < 2\eta$ for all P in E_0,

(2) $I_0(1,P) = T_0(P)$ for all P in E_0 and $I_0(0,P) = P$ for all P in E_0,

(3) $I_0(t,P) = P$ for all (t,P) in $[0,1] \times \{(r,\theta): r = 1 + \eta\}$, and

(4) $I_0(t,\cdot)$ is a homeomorphism of E_0 into E_0 for each $t \in [0,1]$.

PROOF. For each (r,θ) in E_0 define $\tau_\theta(r) = \frac{1}{2} r + \frac{\eta}{2} + \frac{1}{2}$ and let $T_0(r,\theta) = (\tau_\theta(r),\theta)$. Now define the homotopy I_0 by $I_0(t,(r,\theta)) = \bigl(r + t(\tau_\theta(r) - r),\theta\bigr)$. Clearly T_0 and I_0 have the required properties. $\square$

LEMMA 2.13. There is a homeomorphism T_1: $E_1 \to E_1 \cap \{(r,\theta): r \ge 1\}$ and a homotopy I_1: $[0,1] \times E_1 \to E_1$ which have the following properties:

(1) $T_1(g(P_0)) = P_1$,

(2) $T_1(P) = T_0(P)$ for all P in $E_0 \cap E_1$,

(3) $T_1(P) = P$ for all P in $\partial E_1 \cap \{(r,\theta): r \ge \eta + 1\}$,

(4) $I_1(1,P) = T_1(P)$ for all P in E_1, and $I_1(0,P)$ for all P in E_1,

(5) $I_1(t,P) = P$ for all P in $\partial E_1 \cap \{(r,\theta): r \ge 1 + \eta\}$,

(6) $I_1(t,P) = I_0(t,P)$ for all P in $E_0 \cap E_1$, and

(7) $I_1(t,\cdot)$ is a homeomorphism of E_1 into E_1 for each $t \in [0,1]$.

PROOF. There is a homeomorphism ϕ which maps E_1 onto $\{(r,\theta): 1 - \eta \le r \le 1 + 3\eta,\ \theta_1 \le \theta \le \theta_2\}$ where θ_1 and θ_2 are so that $|\theta_2 - \theta_1| < 2\pi$, $w_1 = (1,\theta_1)$ and $w_2 = (1,\theta_2)$, and which has the additional properties, (1) ϕ is the identity on $E_1 \cap \{(r,\theta): 1 - \eta \le r \le 1 + \eta\}$ and (2) $\phi(P_1) = (1 + 2\eta,\theta_0)$ where $g(P_0) = (1,\theta_0)$. Now let

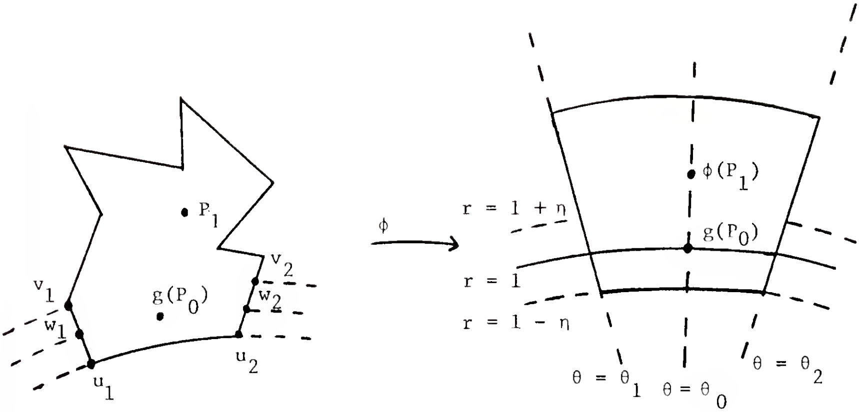

Figure 12. The map ϕ.

$$\tau_{\theta_1}(r) = \begin{cases} \frac{1}{2} r + \frac{\eta}{2} + \frac{1}{2} & \text{if } 1 - \eta \le r \le 1 + \eta \\ r & \text{if } 1 + \eta < r \le 1 + 3\eta, \end{cases}$$

and let $\tau_{\theta_2}(r) = \tau_{\theta_1}(r)$. Now let

$$\tau_{\theta_0}(r) = \begin{cases} 2r + 2\eta - 1 & \text{if } 1 - \eta \le r \le 1 \\ \frac{1}{3} r + \frac{2}{3} + 2\eta & \text{if } 1 \le r \le 1 + 3\eta. \end{cases}$$

For θ between θ_1 and θ_0 there is some λ, $0 < \lambda < 1$ so that $\theta = \lambda\theta_1 + (1 - \lambda)\theta_0$. Let $\tau_\theta(r) = \lambda\tau_{\theta_1}(r) + (1 - \lambda)\tau_{\theta_0}(r)$. For θ between θ_0 and θ_2 there is some λ, $0 < \lambda < 1$ so that $\theta = \lambda(\theta_0) + (1 - \lambda)\theta_2$. Let $\tau_\theta(r) = \lambda\tau_{\theta_0}(r) + (1 - \lambda)\tau_{\theta_1}(r)$. Now let $\tau(r,\theta) = (\tau_\theta(r),\theta)$ and let $T_1 = \phi^{-1}\tau\phi$. For each $t \in [0,1]$ let $\Phi(t,(r,\theta)) = \big(r + t(\tau_\theta(r) - r),\theta\big)$ and then let $I_1(t,P) = \phi^{-1}\Phi_t\phi(P)$. Clearly T_1 and I_1 have the required properties. $\square$

PROPOSITION 2.14. There is a homeomorphism $T: \{(r,\theta): r \ge 1 - \eta\} \to \{(r,\theta): r \ge 1\}$ and a homotopy

I: $[0,1] \times \{(r,\theta): r \geq 1 - \eta\} \to \{(r,\theta): r \geq 1 - \eta\}$ with the following properties:

(1) $T(g(P_i)) = P_{i+1}$ for $0 \leq i \leq N - 1$,

(2) $T(D_j) = D_j$ for $1 < j \leq N$, and $T(E_0) \subset E_0$, $T(E_1) \subset E_1$,

(3) $T(P) = P$ for all P in $\{(r,\theta): r \geq 1 - \eta\} \setminus \left(\bigcup_{j=2}^{N} D_j \cup E_0 \cup E_1\right)$,

(4) $I(1,P) = T(P)$ for all P in $\{(r,\theta): r \geq 1 - \eta\}$ and $I(0,P) = P$ for all P in $\{(r,\theta): r \geq 1 - \eta\}$,

(5) $I(t,P) = P$ for all P in $\{(r,\theta): r \geq 1 - \eta\} \setminus \left(\bigcup_{j=1}^{k} D_j \cup E_0 \cup E_1\right)$,

(6) I_t is a homeomorphism of $\{(r,\theta): r \geq 1 - \eta\}$ into itself for each $t \in [0,1]$, and

(7) $I_t g: A \to E^2$ has exactly the same fixed points as g has, for each $t \in [0,1]$.

PROOF. By lemma 2.9 for each j, $2 \leq j \leq N$ there is a homeomorphism T_j and a homotopy I_j with the properties that

(1) $T_j(g(P_{j-1})) = P_j$,

(2) $T_j(P) = P$ for all P in ∂D_j,

(3) $I_j(1,P) = T_j(P)$ and $I_j(0,P) = P$ for all P in ∂D_j,

(4) $I_j(t,P) = P$ for all (t,P) in $[0,1] \times \partial D_j$, and

(5) $I_j(t,P)$ is a homeomorphism of $\overline{D_j}$ onto $\overline{D_j}$ for each t in $[0,1]$.

Lemmas 2.12 and 2.13 construct the homeomorphisms T_0 and T_1 and the homotopies I_0 and I_1 for E_0 and E_1 respectively, and list their properties. Now define the homeomorphism T from $\{(r,\theta): r \geq 1 - \eta\}$ into $\{(r,\theta): r \geq 1\}$ by

$$T(P) = \begin{cases} T_j(P) & \text{if } P \in D_j \quad 2 \leq j \leq N \\ T_i(P) & \text{if } P \in E_i \quad i = 1 \text{ or } 2 \\ P & \text{if } P \in \{(r,\theta): r \geq 1 - \eta\} \setminus \left(\bigcup_{j=2}^{N} D_j \cup E_0 \cup E_1\right). \end{cases}$$

Define the homotopy I by

$$I(t,P) = \begin{cases} I_j(t,P) & \text{if } P \in D_j \quad 2 \le j \le N \\ I_i(t,P) & \text{if } P \in E_i \quad i = 0 \text{ or } 1 \\ P & \text{if } P \in \{(r,\theta)\colon r \ge 1-\eta\} \setminus \left(\bigcup_{j=2}^{N} D_j \cup E_0 \cup E_1\right). \end{cases}$$

The first six properties are immediate from the construction of the T_j's and the I_j's and the fact that the sets $D_2, \ldots D_N$, and $E_0 \cup E_1$ are disjoint. If F is a fixed point of g, then $F \in \{(r,\theta)\colon r \ge 1-\eta\} \setminus \left(\bigcup_{j=2}^{\infty} D_j \cup E_0 \cup E_1\right)$; since $F \notin M_j$ and $D_j \subseteq M_j$ and $E_1 \subseteq M_1$, F cannot be in D_j or in E_1 and by choice of η, $F \notin E_0$. So $I_t(g(F)) = T_t(F) = F$. Suppose P is not a fixed point of g. If $g(P) \in \{(r,\theta)\colon r \ge 1-\eta\} \setminus \left(\bigcup_{j=2}^{N} D_j \cup E_0 \cup E_1\right)$, then $I_t(g(P)) = g(P)$ so P is not a fixed point of $I_t \cdot g$. If $g(P) \in D_j$, then $P \notin D_j$, and then $I_t(g(P)) \in D_j$ implies $I_t \cdot g$ does not fix P. If $g(P) \in E_1$, then $P \notin E_1$, and then $I_t(g(P)) \in E_1$ implies $I_t \cdot g$ does not fix P. If $g(P) \in E_0$, then $d(P,g(P)) > 2\eta$ and $d(g(P), I_t g(P)) < 2\eta$ imply that $P \ne I_t(g(P))$. So $I_t \cdot g$ has exactly the same fixed points as g on A. □

LEMMA 2.15. There is a continuous function $H\colon [0,1] \times E^2 \to E^2$ with the following properties:

(1) $H_0(P) = h(P)$ for all P in E^2,

(2) $\Pi \cdot H_1(P) = (T \cdot g) \cdot \Pi$ on $\{(x,y)\colon y \ge -\eta\}$, and

(3) for each $t \in [0,1]$ H_t is a homeomorphism of E^2 onto itself which has exactly the same fixed points as h.

PROOF. There is a continuous map $I\colon [0,1] \times \{(x,y)\colon y \ge -\eta\} \to \{(x,y)\colon y \ge 0\}$ so that $I \cdot (id \times \Pi) = \Pi \cdot I$, that is to say, so that I makes the following diagram commute:

$$\begin{array}{ccc} [0,1] \times \{(x,y)\colon y \geq -\eta\} & \xrightarrow{\ I\ } & \{(x,y)\colon y \geq 0\} \\ \Big\downarrow{\scriptstyle \mathrm{id} \times \Pi} & & \Big\downarrow{\scriptstyle \Pi} \\ [0,1] \times \{(r,\theta)\colon r \geq 1-\eta\} & \xrightarrow{\ \mathrm{I}\ } & \{(r,\theta)\colon r \geq 1\} \end{array},$$

and so that $I(0,P) = P$ for all points P in $\{(x,y)\colon y \geq -\eta\}$. In this case $I(1,P) = P$ for all points P in $\{(x,y)\colon y \geq -\eta\} \setminus \Pi^{-1}\Big(\bigcup_{j=2}^{N} D_j \cup E_0 \cup E_1\Big)$. Extend I to $[0,1] \times E^2$ by requiring that $I(t,(x,y)) = (x,y + t\eta)$. Now let $H(t,P) = I(t,h(P))$ for all $(t,P) \in [0,1] \times E^2$. So $H(0,P) = I(0,h(P)) = h(P)$ for all P in E^2, and $\Pi\cdot H(t,P) = I(t,g(P))\cdot\Pi$ on $\{(x,y)\colon y \geq -\eta\}$ for all t in $[0,1]$ so that $\Pi\cdot H(1,P) = I(1,g(P))\cdot\Pi = (T\cdot g)\cdot\Pi$ on $\{(x,y)\colon y \geq -\eta\}$ and since $I_t\cdot g$ has exactly the same fixed points as g, H_t has exactly the same fixed points as h. □

Some Index Lemmas and the Curve C*

In this section we prove several lemmas about the index along simple arcs with one end point in $y \leq 0$ and the other in $y \geq \Gamma(x)$, and construct the simple curve C*.

LEMMA 2.16. Suppose $f \in \{H_t\colon 0 \leq t \leq 1\}$. If (x,y) is in $y \leq 0$ and $(x',y') = f(x,y)$, then $x' > x$, $y' \geq y$, and $x' - x > 2(y' - y)$. If (x,y) is in $y \geq \Gamma(x)$ and $(x',y') = f(x,y)$, then $x' < x$.

PROOF. This all follows by the choice of $\delta(x) < \mu_3$ and the choice of η to satisfy condition (6) in lemma 2.11 and the construction of the homeomorphism T and the homotopy I. □

LEMMA 2.17. Suppose $C\colon [0,1] \to E^2$ is a simple arc with $C(0)$ in $y \leq 0$ and $C(1)$ in $y \geq \Gamma(x)$, which contains no fixed point of h. Let θ_1^t be the angle from $\overrightarrow{C(0),\ C(0) + (1,0)}$ to $\overrightarrow{C(0),\ H_t(C(0))}$ and let θ_2^t

be the angle from $\overline{C(1),\ C(1) + (1,0)}$ to $\overline{C(1),\ H_t(C(1))}$ then

$\mathrm{Ind}_{H_t} C = (\theta_2^t - \theta_1^t)/2\pi \pmod 1$ and $\frac{1}{8} < \mathrm{Ind}_{H_t} C < \frac{3}{4} \pmod 1$.

PROOF. Since $\theta_2^t - \theta_1^t$ is equal to the angle between $\overline{D(C(0),\ H_t(C(0)))}$ and $\overline{D(C(1),\ H_t(C(1)))}$, $\mathrm{Ind}_{H_t} C = (\theta_2^t - \theta_1^t)/2\pi \pmod 1$.

Since H_t moves all points of $y \leq 0$ to the right and since $\eta < \frac{1}{2} \min d(P,h(P))$ for all P in $y = 0$, $D(C(0),H_t(C(0)))$ is between $(1,0)$ and $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ counterclockwise on S^1. Since H_t moves all points of $y \geq \Gamma(x)$ to the left, $D(C(1),\ H_t(C(1)))$ is strictly between $(0,1)$ and $(0,-1)$ counterclockwise on S^1. So the angle θ between $D(C(0),f(C(0)))$ and $D(C(1),H_t(C(1)))$ is strictly between $\frac{\pi}{4}$ and $\frac{3\pi}{2}$; since $\mathrm{Ind}_{H_t}(C) = \frac{\theta}{2\pi} \pmod 1$, $\frac{1}{8} < \mathrm{Ind}_{H_t} C < \frac{3}{4} \pmod 1$. □

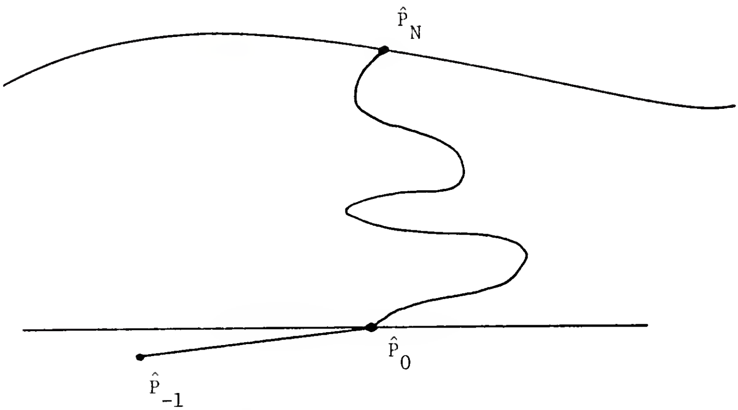

Figure 13. The curve C*.

LEMMA 2.18. Suppose $f \in \{H_t : t \in [0,1]\}$. Let W be a point in $y \leq 0$ and B be a point in $y \geq \Gamma(x)$. Let $\mathcal{C} = \{C\colon [0,1] \to E^2\colon C$ is a simple arc with $C(0) = W$, $C(1) = B$, and there is no fixed point of f on $C\}$, then $\mathrm{Ind}_f C$ has the same value for all $C \in \mathcal{C}$.

PROOF. There are two cases, either f is fixed point free or else $\Pi^{-1}(F) = \{(\bar{x} + 2N\pi, \bar{y})$: N is an integer$\}$ is the set of fixed points of f. In the first case, if C and C' are two curves from $\mathcal{C}$, then there is a continuous family of curves C_r, $0 \le r \le 1$ so that $C_0 = C$, $C_1 = C'$, and for all r, $C_r(0) = W$, $C_r(1) = B$, and $\mathrm{Ind}_f C_r$ is defined for all r since f has no fixed points. Therefore $\mathrm{Ind}_f C = \mathrm{Ind}_f C'$.

So suppose $\Pi^{-1}(F)$ is the set of fixed points of f. First suppose α is a simple closed curve, oriented counterclockwise, which lies in $\mathrm{int}\tilde{A}$, contains a single fixed point $(\bar{x},\bar{y})$ of f in its interior and is contained in $x_1 < x < x_1 + 2\pi$ where x_1 is some number satisfying $\bar{x} - 2\pi < x_1 < \bar{x}$. Let C_1: $[0,1] \to E^2$ be the vertical line segment from $(x_1,\Gamma(x_1))$ to $(x_1,0)$; let C_2: $[1,2] \to E^2$ be the horizontal line segment from $(x_1,0)$ to $(x_1 + 2\pi,0)$; let C_3: $[2,3] \to E^2$ be the vertical line segment from $(x_1 + 2\pi,0)$ to $(x_1 + 2\pi,\Gamma(x_1 + 2\pi))$; and let C_4: $[3,4] \to E^2$ be the graph of $y = \Gamma(x)$ from $(x_1 + 2\pi,\Gamma(x_1 + 2\pi))$ to $(x_1,\Gamma(x_1))$. Since $D((x_1,y),f(x_1,y))$ is the same as $D((x_1,y) + (2\pi,0),f(x_1,y) + (2\pi,0))$ which is the same as $D((x_1 + 2\pi,y),\ f(x_1 + 2\pi,y))$, $\mathrm{Ind}_f C_1 = \mathrm{Ind}_f {-C_3}$. Since $D((x,\Gamma(x),\ f(x,\Gamma(x)))$ lies strictly between (0,1) and (0,-1) c.c. for all x, $-\frac{1}{2} < \mathrm{Ind}_f C_4 < \frac{1}{2}$ and because $D((x_1,\Gamma(x_1)),f(x_1,\Gamma(x_1))$ $= D((x_1 + 2\pi,\Gamma(x_1 + 2\pi)),f(x_1 + 2\pi,\Gamma(x_1 + x_1 + 2\pi))$, $\mathrm{Ind}_f C_4 = 0 \pmod 1$. Hence $\mathrm{Ind}_f C_4 = 0$. Since $D((x,0),\ f(x,0))$ lies strictly between (1,0) and $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ c.c. for all x, $-\frac{1}{4} < \mathrm{Ind}_f C_2 < \frac{1}{4} \pmod 1$, and because $D((x_1,0),f(x_1,0)) = D((x_1 + 2\pi,0),\ f(x_1 + 2\pi,0))$, $\mathrm{Ind}_f C_2 = 0 \pmod 1$, hence $\mathrm{Ind}_f C_2 = 0$. So $\mathrm{Ind}_f C_1C_2C_3C_4 = \mathrm{Ind}_f C_1 + \mathrm{Ind}_f C_2 + \mathrm{Ind}_f C_3 + \mathrm{Ind}_f C_4 = 0$. Since α is homotopic to $C_1C_2C_3C_4$ via a homotopy whose range lies in $\overline{\mathrm{int}C_1C_2C_3C_4} \setminus \mathrm{int}\alpha$, $\mathrm{Ind}_f\alpha = \mathrm{Ind}_f C_1C_2C_3C_4$, hence $\mathrm{Ind}_f\alpha = 0$.

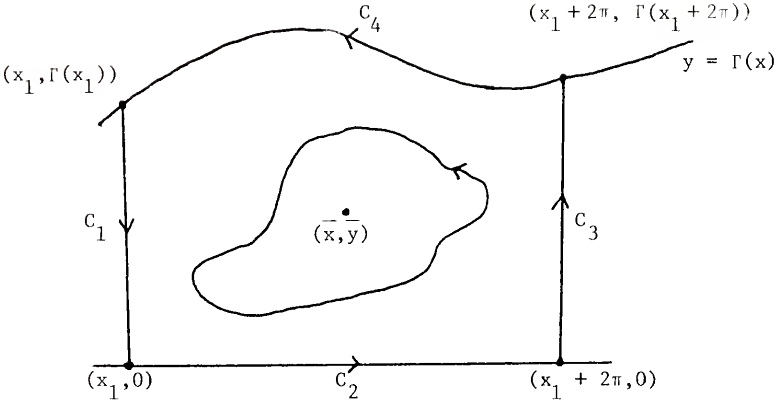

Figure 14. The curves $C_1C_2C_3C_4$ and α.

Now suppose that α_1 and α_2 are simple arcs from W to B which intersect only at W and B, and there is exactly one fixed point $(\bar{x}, \bar{y})$ of f in $\mathrm{int}(\alpha_1 \cup \alpha_2)$. There is a positive number ε so that $\overline{N_\varepsilon(\bar{x}, \bar{y})} \subseteq \mathrm{int}(\alpha_1 \cup \alpha_2)$. Let $\beta = \partial N_\varepsilon(\bar{x}, \bar{y})$ oriented counterclockwise and let p and q be two points on β. There is a simple arc ψ_1 from W to p which lies in $\mathrm{int}(\alpha_1 \cup \alpha_2) \setminus \overline{N_\varepsilon(\bar{x}, \bar{y})}$ except for W and p, and a simple arc ψ_2 from q to B which lies in $\mathrm{int}(\alpha_1 \cup \alpha_2) \setminus \overline{N_\varepsilon(\bar{x}, \bar{y})}$ except for q and B. Let β_2 be the subarc of β from p to q which has the property that $(\bar{x}, \bar{y}) \in \mathrm{ext}(\psi_1 \cup \beta_2 \cup \psi_2 \cup \alpha_2)$. Let $\beta_1 = \overline{\beta \setminus \beta_2}$ from p to q. Now there is a homotopy from α_2 to $\psi_1 \cup \beta_2 \cup \psi_2$ which fixes W and B and whose range lies in $\overline{\mathrm{int}(\psi_1 \cup \beta_2 \cup \psi_2 \cup \alpha_2)}$; hence $\mathrm{Ind}_f\alpha_2 = \mathrm{Ind}_f\psi_1\beta_2\psi_2$. Similarly, $\mathrm{Ind}_f\alpha_1 = \mathrm{Ind}_f\psi_1\beta_1\psi_2$. However by the previous argument $\mathrm{Ind}_f\beta = 0$; so $\beta = \beta_2 - \beta_1$ implies $\mathrm{Ind}_f\beta_1 = \mathrm{Ind}_f\beta_2$. Hence $\mathrm{Ind}_f\psi_1\beta_1\psi_2 = \mathrm{Ind}_f\psi_1\beta_2\psi_2$ and therefore $\mathrm{Ind}_f\alpha_1 = \mathrm{Ind}_f\alpha_2$.

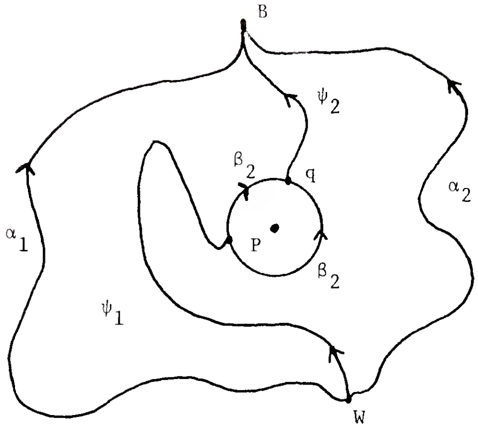

Figure 15. The arcs α_1, α_2, ψ_1, ψ_2, β_1, and β_2.

Now suppose C and C' are two simple arcs in $\mathcal{C}$. Let $K = C \cup C' \cup O$ where O is the union of the bounded components of $E^2 \setminus (C \cup C')$. The set K is compact and every point of ∂K is accessible from ∞. So there is a simple arc ψ from W to B which lies in $E^2 \setminus K$ except for W and B, and which does not contain any fixed point of f. Let P be a point in ψ different from W and B and let Z be a point in C different from W and B. The simple closed curve $\psi \cup C$ has only finitely many of the fixed points of f in its interior, say $Q_1, Q_2, \ldots, Q_k$. By induction a simple arc $Q\colon [0,k] \to E^2$ can be constructed so that Q lies in the interior of $\psi \cup C$ except for $Q(0) = P$ and so that $Q(i) = Q_i$ for $1 \le i \le k$. Further, Q may be extended to a simple arc $\hat{Q}\colon [0,k+1] \to E^2$ by letting $\hat{Q}\colon [k,k+1] \to E^2$ be a simple arc in $\mathrm{int}(\psi \cup C) \setminus Q$ from Q_k to Z. The union α of the arc $\hat{Q}$, the subarc of ψ from W to P and the subarc of C from Z to W is a simple closed curve with B in its exterior. The union β of the arc $\hat{Q}$, the subarc of ψ from P to B and the subarc of C from

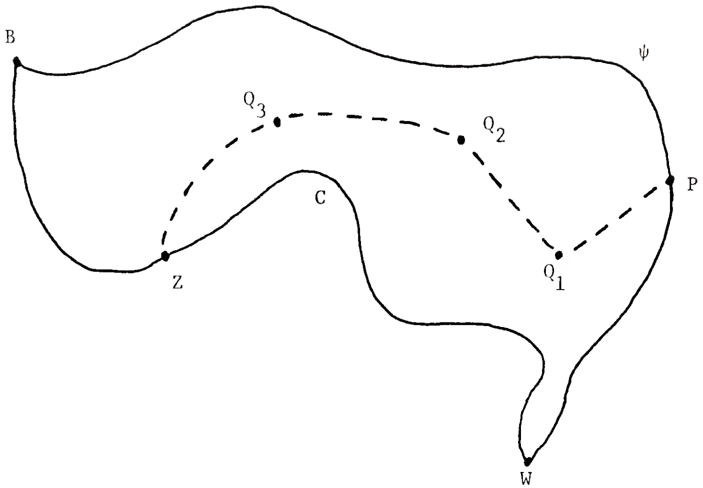

Figure 16. The simple closed curve $\psi \cup C$.

B to Z is a simple closed curve with W in its exterior. By induction there are simple arcs C_i, $1 \le i \le k$ from W to $Q(k - i + \frac{1}{2})$ in $\operatorname{int}\alpha$ except for A and $Q(k - i + \frac{1}{2})$ which intersect only at A, and there are simple arcs C_i^*, $1 \le i < k$ from $Q(k - i + \frac{1}{2})$ to B in $\operatorname{int}\beta$ except for B and $Q(k - i + \frac{1}{2})$ which intersect only at B. Now C, $C_1C_1^*$, $C_2C_2^*$, . . . $C_{k-1}C_{k-1}^*$, ψ is a sequence of arcs from W to B which are pairwise disjoint except for W and B. There is exactly one fixed point of f in the interior of each of the simple closed curves $C \cup C_1C_1^*$, $C_1C_1^* \cup C_2C_2^*$, . . . , $C_{k-2}C_{k-2}^* \cup C_{k-1}C_{k-1}^*$, $C_{k-1}C_{k-1}^* \cup \psi$. Hence by the previous argument $\operatorname{Ind}_f C = \operatorname{Ind}_f C_1C_1^*$, $\operatorname{Ind}_f C_iC_i^* = \operatorname{Ind}_f C_{i+1}C_{i+1}^*$ for $1 \le i < k - 1$ and $\operatorname{Ind}_f C_{k-1}C_{k-1}^* = \operatorname{Ind}_f \psi$. So $\operatorname{Ind}_f C = \operatorname{Ind}_f \psi$. By the same argument $\operatorname{Ind}_f C' = \operatorname{Ind}_f \psi$ and so $\operatorname{Ind}_f C = \operatorname{Ind}_f C'$. $\square$

Let $G_0 = \{(x,y): -\eta < y < 0\}$, and for each integer i let $G_i = H_1^i(G_0)$ where η is as in lemma 2.11. The following easy lemma will be used to show that a certain arc is simple.

LEMMA 2.19. If $i \neq j$, then $G_i \cap G_j = \phi$, and if $|i - j| > 1$, then $\overline{G_i} \cap \overline{G_j} = \phi$.

PROOF. If i is a negative integer, then $G_i = H_1^i(G_0) = \{(x,y): \eta(i - 1) < y \leq \eta(i)\}$. So if i and j are negative integers and $i \neq j$, then $G_i \cap G_j = \phi$; if $|i - j| > 1$, then $\overline{G_i} \cap \overline{G_j} = \phi$. If i and j are any integers so that if $i \neq j$, then $H_1^{-|i| - |j|}(G_i \cap G_j) = G_{i-|i|-|j|} \cap G_{j-|i|-|j|} = \phi$; so $G_i \cap G_j = \phi$; and if $|i - j| > 1$, then $H_1^{-|i| - |j|}(\overline{G_i} \cap \overline{G_j}) = \overline{G}_{i-|i|-|j|} \cap \overline{G}_{j-|i|-|j|} = \phi$; so $\overline{G_i} \cap \overline{G_j} = \phi$. □

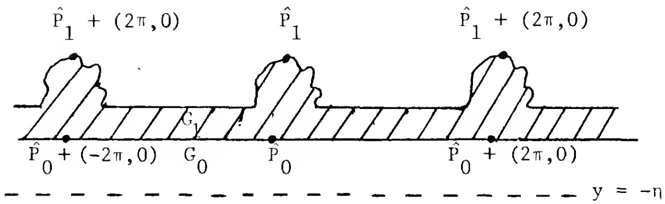

Figure 17. The sets G_0 and G_1.

DEFINITION 2.20. Let $\hat{P}_0 \in \Pi^{-1}(P_0)$, and for each i, $-1 \leq i \leq N + 1$, let $\hat{P}_i = H_1^i(\hat{P}_0)$. So for $0 < i \leq N + 1$, $\hat{P}_i = H_1(\hat{P}_{i-1})$. Note also that $\Pi(\hat{P}_i) = P_i$ for $0 \leq i \leq N$.

DEFINITION 2.21. Let $C_0: [-1,0] \to E^2$ be the straight line segment from $\hat{P}_{-1}$ to $\hat{P}_0$. For each positive integer k define $C_k: (k-1,k] \to E^2$ by

$C_k(t) = H_1^k(C_0(t-k))$ for all $t \in (k-1,k]$. Now let $C: [-1,N+1] \to E^2$ be the union of the maps C_k for $0 \le k \le N+1$.

LEMMA 2.22. The arc $C: [-1,N+1] \to E^2$ is a simple arc with $C(k) = \hat{P}_k$ for each integer k, $-1 \le k \le N$, and $C(t+1) = H_1(C(t))$ for each t in $[-1,N]$.

PROOF. The simple arc C_0 is a subset of G_0, and for each k, $0 < k \le N$, C_k is a subset of G_k so that $C_k \cap C_{k'} \subseteq G_k \cap G_{k'} = \phi$ if $k \ne k'$. If $k' = k+1$, then $\overline{C_k} \cap \overline{C_{k'}} = \hat{P}_k$, and if $|k-k'| > 1$, then $\overline{C_k} \cap \overline{C_{k'}} \subseteq \overline{G_k} \cap \overline{G_{k'}} = \phi$. Hence $C = \bigcup_{k=0}^{N+1} C_k$ is a simple arc. Since $C(k) = C_k(k) = H_1^k(C_0(0)) = H_1^k(\hat{P}_0)$, $C(k) = \hat{P}_k$ for each k, $-1 \le k \le N$. And finally if $t \in (k-1,k]$, then $t+1 \in (k,k+1]$ so that $C(t+1) = H_1^{k+1}\big(C_0(t+1-(k+1))\big) = H_1 H_1^k(C_0(t-k)) = H_1(C(t))$. $\square$

Now let Q be the first point of $C([-1,N])$ which is also a point of $y = \Gamma(x)$.

LEMMA 2.23. There is a number q, $N-1 < q \le N$ so that $C(q) = Q$.

PROOF. Since $\hat{P}_N$ is on $y = \Gamma(x)$, $q \le N$. Since $G_1 = H_1\{(x,y): y \le 0\} \setminus \{(x,y): y \le 0\}$, $\Pi(G_1) \subset M_1$ by the choice of η in lemma 2.11 to satisfy (4). Also since $G_{k+1} = H_1(G_k)$, we have $\Pi(G_{k+1}) = \Pi \cdot H_1(G_k) = T \cdot g \cdot \Pi(G_k) \subseteq T \cdot g(M_k) \subseteq M_{k+1}$ by induction, where $M_k = \{P \in A: P = P_k$ for some δ-chain $P_0, P_1, \ldots, P_k\}$. Since $P_0, P_1, \ldots, P_N$ is a minimal δ-chain $M_k \cap \gamma = \phi$ for $0 \le k \le N-1$, hence $G_k \cap \{(x,y): y = \Gamma(x)\} = \phi$ for $0 \le k \le N-1$, so that $C_k \cap \{(x,y): y = \Gamma(x)\} = \phi$ for $0 \le k \le N-1$. Therefore $q > N-1$. $\square$

Let $C^*: [-1,q] \to E^2$ be the subarc of C from $\hat{P}_{-1}$ to Q. By lemma 2.17, $\frac{1}{8} < \mathrm{Ind}_f C^* < \frac{3}{4} \pmod 1$ for all $f \in \{H_t: 0 \le t \le 1\}$. Since $\Pi(C^* \cap \tilde{A}) \subset \Pi(\bigcup_{k=1}^{N} G_k) \subset \bigcup_{k=1}^{N} M_k = M_N$, C^* does not contain any fixed points of any $f \in \{H_t: 0 \le t \le 1\}$.

Calculation of $\mathrm{Ind}_h C^*$

In this section a homotopy is constructed which allows us to calculate $\mathrm{Ind}_{H_1} C^*$ and from that $\mathrm{Ind}_h C^*$, and we show that $\frac{1}{4} < \mathrm{Ind}_h C^* < \frac{3}{4}$.

Let $\overline{P}$: $[-1,q] \to S^1$ be given by $\overline{P}(t) = D(C(t), H_1C(t)) = D(C(t), C(t+1))$. Now extend this to a map $\overline{P}_0$: $[-1,2q+1] \to S^1$ by

$$\overline{P}_0(t) = \begin{cases} \overline{P}(t) & -1 \le t \le q \\ \overline{P}(q) & q < t \le 2q+1. \end{cases}$$

Note that $\mathrm{Ind}_{H_1} C = (\tilde{P}_0(2q+1) - \tilde{P}_0(-1))/2\pi$ where $\tilde{P}_0$ is any lift of $\overline{P}_0$, that is, $\tilde{P}_0$ is any map which makes the following diagram commute:

$$\begin{tikzcd} & R \arrow[d, "\Pi"] \\ {[-1,2q+1]} \arrow[ur, dashed, "\tilde{P}_0"] \arrow[r, "\overline{P}_0"'] & S^1 \end{tikzcd}$$

Now any homotopy of the map $\overline{P}_0$ which fixes $\overline{P}_0(-1)$ and $\overline{P}_0(2q+1)$ lifts to a homotopy of $\tilde{P}_0$ which fixes $\tilde{P}_0(-1)$ and $\tilde{P}_0(2q+1)$. More precisely, if $\overline{P}$: $[0,b] \times [-1,2q+1] \to S^1$ is a continuous map with $\overline{P}_\lambda(-1) = \overline{P}_0(-1)$ and $\overline{P}_\lambda(2q+1) = \overline{P}_0(2q+1)$ for all λ in $[0,b]$, then for any lift $\tilde{P}$: $[0,b] \times [-1,2q+1] \to R$ of $\overline{P}$, $\tilde{P}_\lambda(-1) = \tilde{P}_0(-1)$ and $\tilde{P}_\lambda(2q+1) = \tilde{P}_0(2q+1)$ for every λ in $[0,b]$. Hence $\mathrm{Ind}_{H_1} C = (\tilde{P}_\lambda(2q+1) - \tilde{P}_\lambda(-1))/2\pi$ for all $\lambda \in [0,b]$. In lemmas 2.24 through 2.28 such a homotopy will be constructed for $b = q+2$ with the additional property that $\overline{P}_b$ is monotonic and varies less than π, on each of three subintervals whose union is $[-1,2q+1]$, so that $\mathrm{Ind}_{H_1} C$ can be computed from $\overline{P}_b$.

LEMMA 2.24. For $0 \le \lambda \le q + 1$ let

$$\overline{P}_\lambda(t) = \begin{cases} D(C(-1),\ C(t+1)) & -1 \le t \le \lambda - 1 \\ D(C(t-\lambda),\ C(t+1)) & \lambda - 1 \le t \le q \\ D(C(t-\lambda),\ C(q+1)) & q \le t \le q + \lambda \\ D(C(q),\ C(q+1)) & q + \lambda \le t \le 2q + 1. \end{cases}$$

Then $\overline{P}$: $[0,q+1] \times [-1,2q+1] \to S^1$ is a continuous function with $\overline{P}_\lambda(-1) = \overline{P}_0(-1)$ and $\overline{P}_\lambda(2q+1) = \overline{P}_0(2q+1)$ for all λ in $[0, q+1]$.

PROOF. First it is necessary to show that $\overline{P}$ is defined for each (λ,t) in $[0,q+1] \times [-1,2q+1]$. Since C is a simple curve, $t_0 \neq t_1$ implies $C(t_0) \neq C(t_1)$ so that $D(C(t_0), C(t_1))$ is defined if $t_0 \neq t_1$. For each λ and t, $\overline{P}_\lambda(t) = D(C(t_0), C(t_1))$ for some t_0 and t_1 with $t_0 \neq t_1$ since in the first part of the definition of $\overline{P}_\lambda$, $-1 \le t$ so that $-1 < t + 1$ hence $-1 \neq t + 1$, in the second part $t - \lambda \le t - 0 < t + 1$ hence $t - \lambda \neq t + 1$, in the third part $t \le tq + \lambda$ implies $t - \lambda \le q < q + 1$ hence $t - \lambda \neq q + 1$, and in the last part $q \neq q + 1$. The map $\overline{P}$ is continuous since each of the four parts is, and they match up properly. $\square$

Note that the map $\overline{P}_{q+1}$: $[-1,q+1] \to S^1$ is given by

$$\overline{P}_{q+1}(t) = \begin{cases} D(C(-1),\ C(t+1)) & -1 \le t \le q \\ D(C(t-q-1),\ C(q+1)) & q < t \le 2q + 1. \end{cases}$$

In the construction of $\overline{P}$: $[q+1,\ q+2] \times [-1,\ 2q+1] \to S^1$ in lemma 2.28 we will use the simple arcs and the homotopies constructed in the next three lemmas.

LEMMA 2.25. There is a simple arc $\alpha:[-1,q] \to E^2$ so that $\alpha(-1) = \hat{P}_{-1}$, $\alpha(q) = Q$, $H_1(Q)$ is not on α and so that there is a number S, $-1 < S < q$ so that $D(\alpha(t), H_1(Q))$ is monotonic and varies less than π on each of $[-1,S]$ and $[S,q]$.

PROOF. Let ε be a positive number less than each of $\pi/4$, the angle i_1 from $\overline{Q,(x_g, y_{g+1})}$ to $\overline{Q,H_1(Q)}$ and the angle i_2 from $\overline{Q,H_1(Q)}$ to $\overline{Q,(x_g, y_{g-1})}$, where $Q = (x_g, y_g)$. Let L_1 be a ray emanating from Q at an angle $\frac{3\pi}{2} + \varepsilon$, let L_2 be the horizontal line through $\hat{P}_{-1}$ and let $R = (x_r, y_r)$ be the point of intersection of L_1 and L_2. Let $\alpha:[-1,S] \to E^2$ be a parametrization of the closed line segment from $\hat{P}_{-1}$ to R where S is any number between -1 and q, and let α: $[S,q] \to E^2$ be a parametrization of the line segment from R to Q. In case $x_r \geq \hat{x}_{-1}$, $D(\alpha(t), H_1(Q))$ is nondecreasing and in case $x_r < x_{-1}$, $D(\alpha(t), H_1(Q))$ is nonincreasing on $[-1,S]$ and nondecreasing on $[S, q]$. □

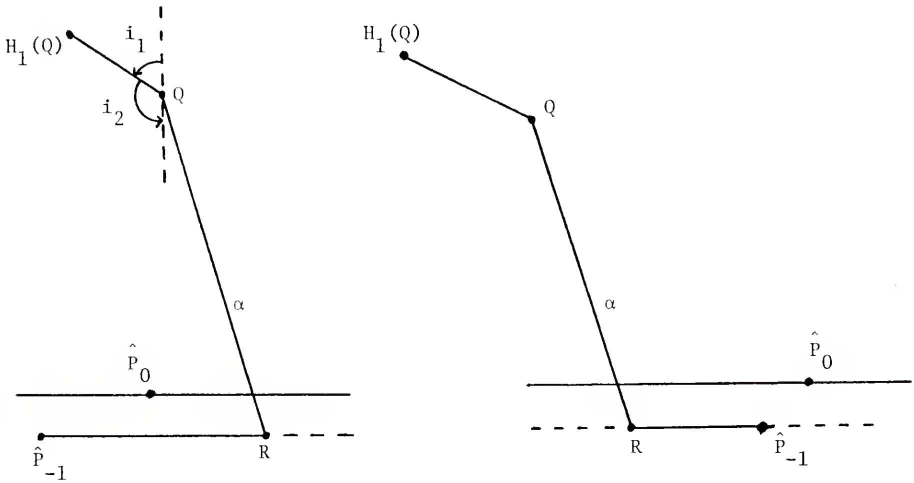

Figure 18. The two possible configurations of α.

LEMMA 2.26. There is a continuous function $N: [0,1] \times [-1,q] \to E^2$ so that $N_0(t) = C(t)$, $N_1(t) = \alpha(t)$, and for all $\mu \in [0,1]$, $N_u(-1) = \hat{P}_{-1}$ and $N_u(q) = Q$ and, moreover, so that $H_1(Q)$ does not lie in the range of N_1.

PROOF. Let $\psi_1 = \{(x,y): x = x_{-1},\ y \le y_{-1}\} \cup \overline{\hat{P}_{-1}\hat{P}_0}$, $\psi_2 = \{(x,y): x = x_q,\ y \ge y_q\}$, and let C' be the subarc of C from P_0 to Q. Finally let $\psi = \psi_1 \cup C' \cup \psi_2$. Each of ψ_1, C', and ψ_2 is a simple arc. Since $\psi_1 \setminus \{P_0\} \subseteq \{(x,y): y < 0\}$ and $C' \subseteq \{(x,y): y \ge 0\}$, $\psi_1 \cap C' = \{P_0\}$. Since $\psi_1 \subseteq \{(x,y): y \le 0\}$ and $\psi_2 \subseteq \{(x,y): y \ge \Gamma(x) > 0\}$, $\psi_1 \cap \psi_2 = \phi$. Finally $\psi_2 \cap C' = \{Q\}$ because otherwise there would be some $w \ne Q$ in $\psi_2 \cap C'$, w would therefore be in $\{(x_q,y): y > \Gamma(x_q)\}$ so the subarc of C' from P_0 to w must cross $y = \Gamma(x)$ contradicting Q being first intersection of C with $y = \Gamma(x)$. Hence ψ is a simple arc. The open set $E^2 \setminus \psi$ has exactly two components, say A_1 and A_2; suppose A_2 is the component which contains $\hat{P}_{-1} + (-1,0)$, that is, A_2 contains the points "to the left of" ψ. Now $H_1(Q)$ is not on C' since C is simple; $H_1(Q)$ is not on ψ_2 since its x-coordinate is smaller than that of Q; and $H_1(Q)$ is not in ψ_1 since its y-coordinate is greater than 0. So $H_1(Q)$ is either in A_1 or in A_2.

Suppose $H_1(Q)$ is in A_1 and consider the set $H_1(\psi_2)$. Now $h(\psi_2)$ is a ray parallel to ψ_2 which lies to the left of ψ_2. Since, by choice of δ, I_1, the homotopy constructed in lemma 2.15, moves points less than $\frac{1}{2}|x_q - x_{q+1}|$, $H_1(\psi_2)$ lies outside $N(\psi_2, \frac{1}{2}|x_q - x_{q+1}|)$ hence does not intersect ψ_2. Furthermore, if $m = \max\Gamma(x) + 1$, then the ray $\{(x,y): x = x_{q+1},\ y \ge m\}$ is fixed by I_1 so that $\{(x,y): x = x_{q+1},\ y \ge m\} \subseteq H_1(\psi_2) \cap A_2$. Since $H_1(\psi_2)$ contains points from each of A_1 and A_2, $H_1(\psi_2) \cap \psi \ne \phi$ and since $H_1(\psi_2) \cap \psi_2 = \phi$ and

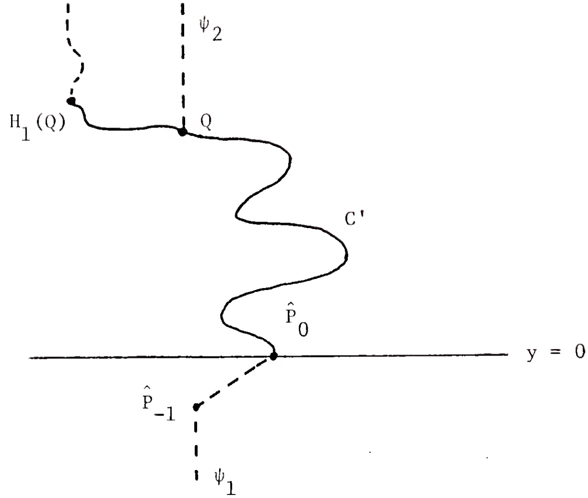

Figure 19. The curve ψ_2, C', and ψ_1.

$H_1(\psi_2) \cap \psi_1 = \phi$, $H_1(\psi_2) \cap C' \neq \phi$. If $z \in H_1(\psi_2) \cap C'$, then $H_1^{-1}(z) \in \psi_2 \cap H_1^{-1}(C')$ so that $H_1^{-1}(z)$ is an element in ψ_2 which is on the subarc of C' from $\hat{P}_{-1}$ to z contradicting the fact that Q is only such. Hence $H_1(Q) \notin A_1$; so $H_1(Q) \in A_2$.

Now let $\ell = \max\{y: (x,y) \in C([-1, q + 1])\} + 1$, and let $k = \max\{x: (x,y) \in C([-1, q + 1])\} + 1 + |x_r - x_0|$, where $R = (x_r, y_r)$ as in the previous lemma. Let $\beta: [-1, q] \to E^2$ be a parametrization of the union of the straight line segments from $\hat{P}_{-1}$ to $(x_{-1}, -2\eta)$, from $(x_{-1}, -2\eta)$ to $(k, -2\eta)$, from $(k, -2\eta)$ to (k,ℓ), from (k,ℓ) to (x_q,ℓ), and from (x_q,ℓ) to Q. By choice of k and ℓ, $\beta \cup C'$ is a simple closed curve, and $\beta \cup \mathrm{int}(\beta \cup C') \subseteq A_1$ so that $H_1(Q) \notin \beta \cup \mathrm{int}(\beta \cup C')$, and

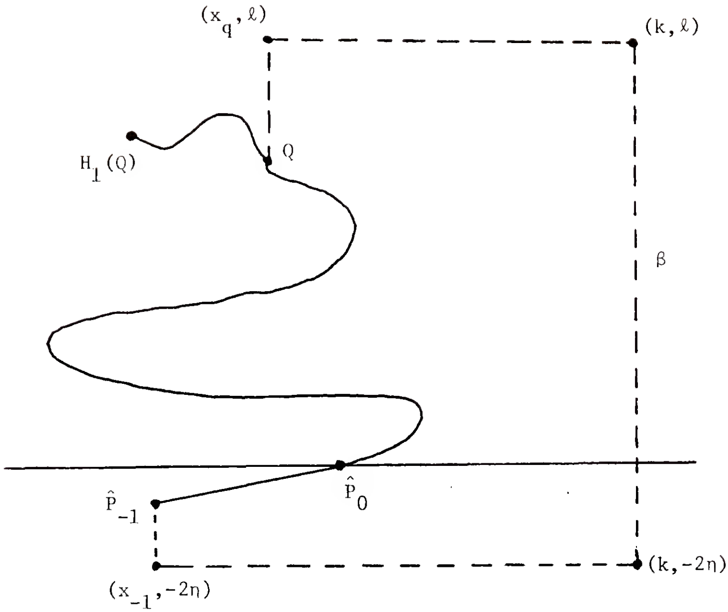

Figure 20. The curve β.

$H_1(Q) \notin C'$ since C is simple; so $H_1(Q)$ lies in the exterior of $\beta \cup C'$. Hence there is a continuous function $G: [0,1] \times [-1,q] \to E^2$, with $G_0(t) = C(t)$, $G_1(t) = \beta(t)$, and for all $\mu \in [0,1]$ $G_u(-1) = \hat{P}_{-1}$ and $G_u(q) = Q$ and whose range lies in $\overline{\mathrm{int}(\beta \cup C')}$. In either case $\alpha \cup \beta$ is a simple closed curve. Since α lies in $\{(x,y)\colon x \geq x_q\} \cup \{(x,y)\colon y \leq 0\}$, $H_1(Q) \notin \alpha$, and since $\mathrm{int}(\alpha \cup \beta) \subseteq \{(x,y)\colon x \geq x_q\} \cup \{(x,y)\colon y \leq 0\}$, $H_1(Q) \notin \overline{\mathrm{int}(\alpha \cup \beta)}$. Hence there is a continuous function $J: [0,1] \times [-1,q] \to E^2$ so that $J_0(t) = \beta(t)$ and $J_1(t) = \alpha(t)$ for all t in $[-1,q]$ and for all $\mu \in [0,1]$, $J_u(-1) = P_{-1}$ and $J_u(q) = Q$ and whose range lies in $\overline{\mathrm{int}(\alpha \cup \beta)}$. Now define $N: [0,1] \times [-1,q] \to E^2$ by

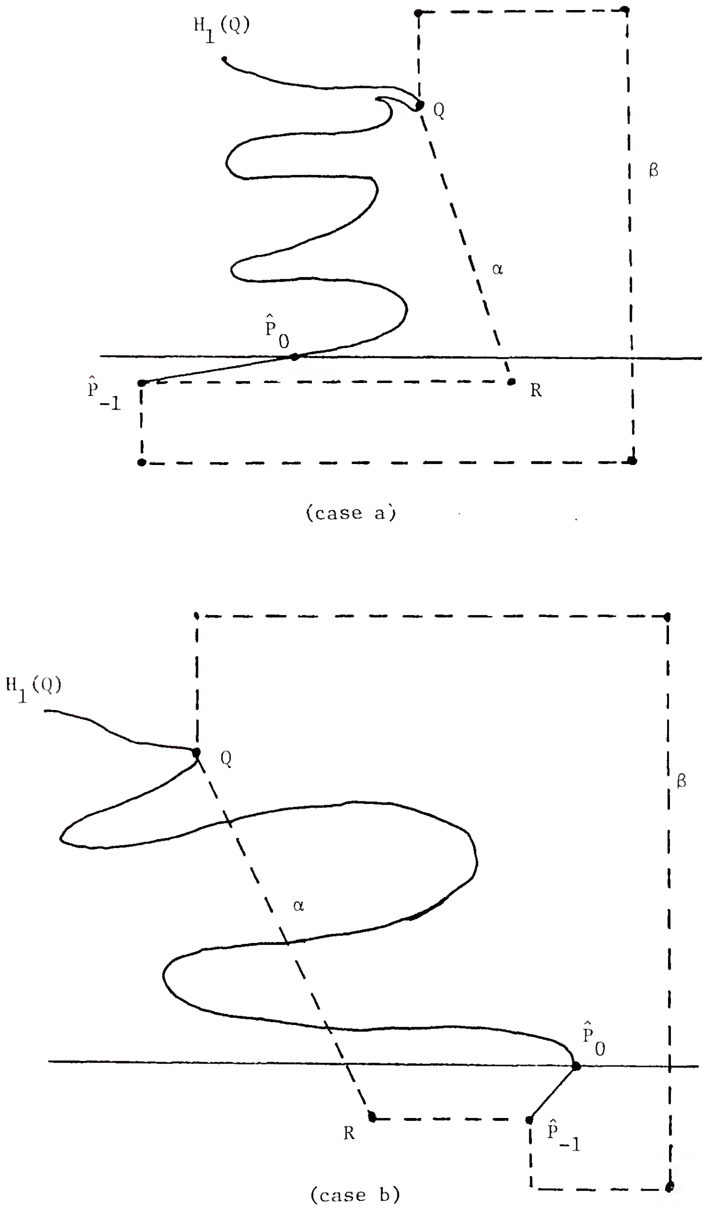

Figure 21. The curves α and β.

$$N_\lambda(t) = \begin{cases} G_{2\lambda}(t) & 0 \le \lambda \le \frac{1}{2} \\ J_{2\lambda-1}(t) & \frac{1}{2} < \lambda \le 1. \end{cases}$$

Clearly N has the required properties. □

LEMMA 2.27. There is a simple arc $\sigma: [0,q+1] \to E^2$ so that $\sigma(0) = \hat{P}_0$, $\sigma(q+1) = H_1(Q)$ and $D(\hat{P}_{-1}, \sigma(t))$ is either increasing, a constant or decreasing, and varies less than π.

PROOF. Let $\sigma: [0,q+1] \to E^2$ be the straight line segment from $\hat{P}_0$ to $H_1(Q)$. If $H_1(Q)$ lies above the straight line L containing $\hat{P}_0$ and $\hat{P}_{-1}$ then $D(\hat{P}_{-1}, \sigma(t))$ is increasing, if $H_1(Q)$ lies on the line L then $D(\hat{P}_{-1}, \sigma(t))$ is constant and if $H_1(Q)$ lies below the line L, then $D(\hat{P}_{-1}, \sigma(t))$ is decreasing. It is clear that $D(\hat{P}_{-1}, \sigma(t))$ varies less than π in either case. □

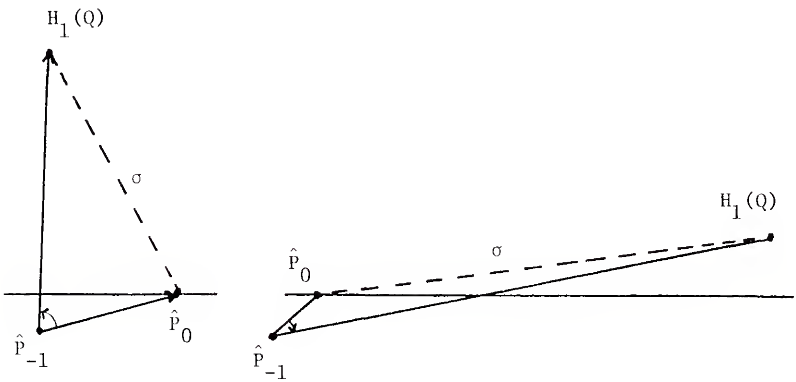

Figure 22. (a) $D(\hat{P}_{-1}, \sigma(t))$ is increasing, (b) $D(\hat{P}_{-1}, \sigma(t))$ is decreasing.

LEMMA 2.28. For $q + 1 \le \lambda \le q + 2$ there is some $\mu \in [0,1]$ so that $\lambda = q + 1 + \mu$. Let

$$\overline{P}_\lambda(t) = \overline{P}_{q+1+\mu}(t) = \begin{cases} D(C(-1), (1-\mu)C(t+1) + \mu\sigma(t+1) & -1 \le t \le q \\ \\ D(N_\mu(t-q-1), C(q+1)) & q < t \le 2q+1, \end{cases}$$

then $\overline{P}$: $[q + 1,\ q + 2] \times [-1,\ 2q + 1] \to S^1$ is a continuous function with $\overline{P}_\lambda(-1) = \overline{P}_0(-1)$ and $\overline{P}_\lambda(2q + 1) = \overline{P}_0(2q + 1)$ for all λ in $[q + 1,\ q + 2]$, and $\overline{P}$ matches up properly with the $\overline{P}_{q+1}$ defined in lemma 2.24.

PROOF. The map $\overline{P}$ is defined for all (λ,t) in $[q+1,\ q+2] \times [-1,2q+1]$ because, in the first part of the definition of $\overline{P}$, $C(-1)$ is never equal to $(1-\mu)C(t+1) + \mu\sigma(t+1)$ since $(1-\mu)C(t+1) + \mu\sigma(t+1) \subseteq \{(x,y)\colon y \ge 0\}$, and in the second part of the definition of $\overline{P}$, $N_\mu(t - q - 1)$ is never equal to $C(q + 1)$, since $C(q + 1) = H_1(Q)$ is not in the range of N. The function $\overline{P}$ is continuous since each part is and they are matched up properly. $\square$

LEMMA 2.29. The map $\overline{P}_{q+2}$: $[-1,\ 2q + 1] \to S^1$, which is given by

$$\overline{P}_{q+2}(t) = \begin{cases} D(C(-1), \sigma(t+1)) & -1 \le t \le q \\ \\ D(\alpha(t-q-1),\ C(q+1)) & q < t \le 2q+1 \end{cases}$$

is monotonic, and varies less than π, on each of three subintervals of $[-1,\ 2q + 1]$ whose union is $[-1,\ 2q + 1]$.

PROOF. On $[-1,\ q + 1]$, $\overline{P}_{q+2}(t) = D(C(-1),\ \sigma(t + 1))$ is monotonic with a variation of less than π by lemma 2.27. On $[q + 1,\ 2q + 2]$ $\overline{P}_{q+2}(t) = D(\sigma(t - q - 1),\ C(q + 1))$. By lemma 2.25 there is an S, $-1 < S < q$ so that $D(\sigma(t - q - 1),\ C(q + 1))$ is monotonic with a

variation of less than π on each of $[q+1,\ q+1+S]$ and $[q+1+S,\ 2q+1]$. □

LEMMA 2.30. Let θ_1 be the angle from $\overrightarrow{\hat{P}_{-1}, (x_{-1}+1,\ y_{-1})}$ to $\overrightarrow{\hat{P}_{-1}, \hat{P}_0}$ and let θ_2 be the angle from $\overrightarrow{Q, (x_q+1,\ y_q)}$ to $\overrightarrow{Q, H_1(Q)}$. Then $\mathrm{Ind}_{H_1} C^* = (\theta_2 - \theta_1)/2\pi$.

PROOF. There is a lift $\tilde{P}_{q+2}$ of $\bar{P}_{q+2}$ so that by the remarks preceding lemma 2.24

$$\mathrm{Ind}_{H_1} C^* = \frac{\tilde{P}_{q+2}(2q+1) - \tilde{P}_{q+2}(-1)}{2\pi} .$$

If A and B are two points which are not exactly opposite one another on the unit circle S^1, then define $\hat{d}(A,B)$ to be the directed (shortest) distance from A to B along the curve S^1. By the previous lemma we have the following three equations:

$$\hat{d}(\bar{P}_{q+2}(-1),\ \bar{P}_{q+2}(q)) = \tilde{P}_{q+2}(q) - \tilde{P}_{q+2}(-1),$$

$$\hat{d}(\bar{P}_{q+2}(q),\ \bar{P}_{q+2}(q+1+S)) = \tilde{P}_{q+2}(q+1+S) - \tilde{P}_{q+2}(q), \text{ and}$$

$$\hat{d}(\bar{P}_{q+2}(q+1+S),\ \bar{P}_{q+2}(2q+1)) = \tilde{P}_{q+2}(2q+1) - \tilde{P}_{q+2}(q+1+S).$$

Therefore $\tilde{P}_{q+2}(2q+1) - \tilde{P}_{q+2}(-1) = (\theta_2 - \theta_1)/2\pi$. So $\mathrm{Ind}_{H_1} C^* = (\theta_1 - \theta_2)/2\pi$. □

LEMMA 2.31. The following inequality holds for the curve C^*: $\frac{1}{4} < \mathrm{Ind}_h C^* < \frac{3}{4}$.

PROOF. For each r in $[0,1]$ let θ_1^r be the angle from $\overrightarrow{\hat{P}_{-1}, (x_{-1}+1, y_{-1})}$ to $\overrightarrow{\hat{P}_{-1}, H_r(\hat{P}_{-1})}$ and let θ_2^r be the angle from $\overrightarrow{Q_1, (x_q+1,\ y_q)}$ to $\overrightarrow{Q_1, H_r(Q)}$. Each of θ_1^r and θ_2^r is a continuous function of r and $\mathrm{Ind}_{H_r} C^* = (\theta_2^r - \theta_1^r)/2\pi$ (mod 1) for every r. Since the index is continuous and $\mathrm{Ind}_{H_1} C^* = (\theta_2^1 - \theta_1^1)/2\pi$, $\mathrm{Ind}_{H_r} C^* = (\theta_2^r - \theta_0^r)/2\pi$ for every r, in particular for r = 0. Therefore $\mathrm{Ind}_h C^* = (\theta_2^0 - \theta_1^0)/2\pi$ and since $\theta_1^0 = 0$ and $\frac{\pi}{2} < \theta_2^0 < \frac{3\pi}{2}$, $\frac{1}{4} < \mathrm{Ind}_h C^* < \frac{3}{4}$. □

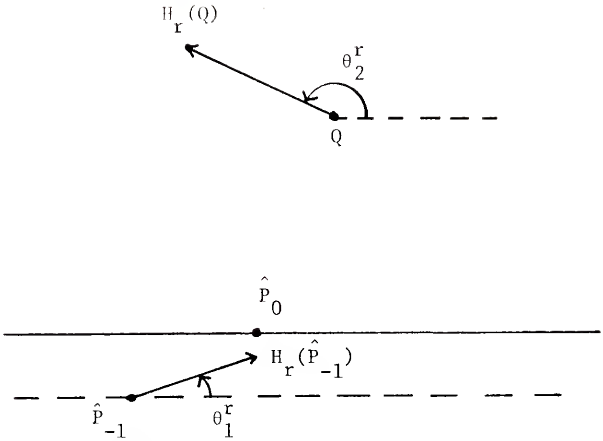

Figure 23. The angles θ_1^r and θ_2^r.

Conclusion of Argument

Now in applying the entire argument, from the construction of δ-chains on, to the homeomorphism g^{-1} which maps $g(A)$ onto A, the only difference would be that the induced map from $g(\tilde{A})$ onto $\tilde{A}$, which in fact can be taken to be h^{-1}, moves points on $y = 0$ to points on $y = 0$ with smaller x-coordinates, and moves points on $h\{(x,y)\colon y = \Gamma(x)\}$ to points on $y = \Gamma(x)$ with larger x-coordinates. Hence in the end we would obtain the following lemma:

LEMMA 2.32. There is a simple curve B which has for one endpoint w in $y \leq 0$ and for its other endpoint $z \in h\{(x,y)\colon y = \Gamma(x)\}$ and which satisfies $-\frac{3}{4} < \mathrm{Ind}_{h^{-1}}B < -\frac{1}{4}$.

Since $-\frac{3}{4} < \mathrm{Ind}_{h^{-1}}B < -\frac{1}{4}$ and $\mathrm{Ind}_{h^{-1}}B = \mathrm{Ind}_h h^{-1}(B)$, $-\frac{3}{4} < \mathrm{Ind}_h h^{-1}(B) < -\frac{1}{4}$. Let $B^* = h^{-1}(B) + (2n\pi, 0)$ for some n sufficiently large so that every point of B^* has greater x-coordinate than every point of C^*. Since h is periodic, $-\frac{3}{4} < \mathrm{Ind}_h B^* < -\frac{1}{4}$. Now C^* is

an arc from W_1 in $y \leq 0$ to Z_1 in $y = \Gamma(x)$ and B^* is a disjoint arc from W_2 in $y \leq 0$ to Z_2 in $y = \Gamma(x)$. Let γ_1 be an arc from W_1 to W_2 in $y \leq 0$ and let γ_2 be the arc along $y = \Gamma(x)$ from Z_2 to Z_1. Since $\mathrm{Ind}_h\gamma_1 = 0$, $\mathrm{Ind}_h C^* = \mathrm{Ind}_h B^*\gamma_2 = \mathrm{Ind}_h B^* + \mathrm{Ind}_h\gamma_2$, by lemma 2.18. Now $-\frac{1}{2} < \mathrm{Ind}_h\gamma_2 < \frac{1}{2}$ so that $-\frac{1}{2} + \frac{-3}{4} < \mathrm{Ind}_h B^* + \mathrm{Ind}_h\gamma_2 < \frac{1}{2} - \frac{1}{4}$ that is $-\frac{5}{4} < \mathrm{Ind}_h B^* + \mathrm{Ind}_h\gamma_2 < \frac{1}{4}$ but $\frac{1}{4} < \mathrm{Ind}_h C^* < \frac{3}{4}$, which is the desired contradiction.

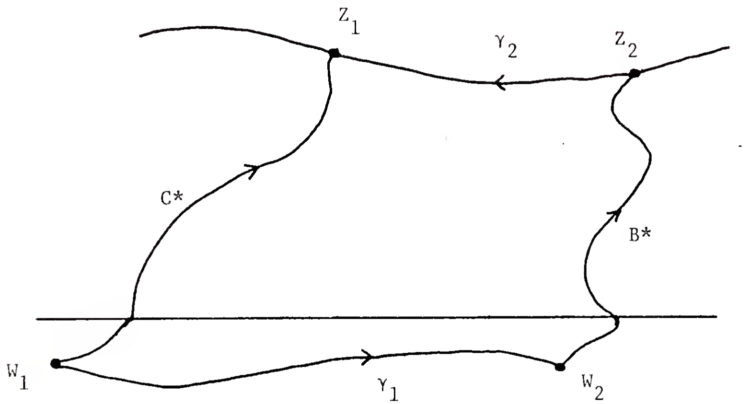

Figure 24. The curves C* and B*.

Directions

The condition in the hypothesis that g be fixed point free on the boundary of A is probably not necessary to the theorem even though it was used in the proof. It could be avoided by constructing the δ-chains in the strip $\tilde{A}$ using the homeomorphism h. However, the lemma on connecting arcs would have to be done with considerably more care.

Another possible improvement of the main theorem would be to replace the curve θ in the conclusion by a curve which misses its image. This could be done if the ring S constructed in the section Constructing δ-chains did not contain the fixed point F in its outer boundary but it is not clear how to accomplish this.

A possible theorem corresponding to the main theorem which concerns itself with the components of the set of fixed points of g would be the following.

Let g be a twist homeomorphism of the annulus A onto g(A), as in the main theorem, then either:

(1) There is a simple closed curve θ so that the annulus bounded by $r = 1$ and θ is mapped onto a proper subset of itself by g or g^{-1},

(2) there is a component of the set of fixed points which separates the components of the boundary of A, or

(3) the fixed point set of g has at least two components.

The analogous theorem for area preserving homeomorphisms was stated and proven by W.D. Neumann in [14]. (In this same paper Neumann studied the periodic points of twist homeomorphisms.)

A related line of investigation concerns the following: What conditions on the twist homeomorphism g insure the existence of an invariant simple closed curve? For a theorem insuring the existence of such curves for area preserving homeomorphisms, under fairly restrictive conditions on g, see Moser [13]. One could also ask: What conditions on the twist homeomorphism g insure the existence of an invariant essential closed curve [a continuum which is the common boundary of the one bounded component and the unbounded component of its complement]?

CHAPTER THREE
AN EXAMPLE

Discussion of Example

Suppose that, as in second chapter, g is a twist homeomorphism of the annulus A bounded by $r = 1$ and the simple closed curve γ, which lies in $r > 1$ and intersects each radial in exactly one point, onto the annulus $g(A)$ bounded by $r = 1$ and the simple closed curve $g(\gamma)$, which also lies in $r > 1$ and intersects each radial in exactly one point. Using the argument indicated by G.D. Birkhoff in his 1925 paper [4], it can be shown that if g has at most one fixed point then there is a ring S having $r = 1$ as a part of its boundary which is mapped onto a proper subset of itself by either g or g^{-1}. In this chapter an example will be given to show that the boundary of the ring S, as constructed by Birkhoff, is not necessarily the union of two simple closed curves. Recall that Birkhoff constructed the ring S, using the homeomorphism g and the continuous function $\delta\colon [0,2\pi] \to R$, in the following way: A sequence of points $P_0, P_1, \ldots, P_n$ is called a δ-chain if P_0 is in $r = 1$ and each point P_{k+1}, for $1 \le k + 1 \le n$, is obtained from $g(P_k) = (r_k, \theta_k)$ by an outward radial motion of less than $\delta(\theta_k)$; that is, $P_{k+1} = (r_k + d_k, \theta_k)$ for some d_k such that $0 \le d_k \le \delta(\theta_k)$. A δ-chain $P_0, P_1, \ldots, P_n$ is terminating if $P_n \notin A$. If g has at most one fixed point F and if δ is chosen to satisfy certain requirements,then either there is no terminating δ-chain for g or else there is no terminating δ-chain for g^{-1},

so that S can be constructed in the first case using g or in the second using g^{-1}. Suppose, without loss of generality, that there is no terminating δ-chain for g. Then the set M_n is given by $M_n = \{P: P = P_n$ for some δ-chain $P_0, P_1, \ldots, P_n\}$ and the ring S is defined to be the complement of the closure of the unbounded component of the complement of the closure of $M = \bigcup_{n=1}^{\infty} M_n$, minus the interior of the unit disk. A twist homeomorphism g from the annulus $1 \le r \le 3\frac{1}{20}$ onto itself with exactly one fixed point F will be constructed and a continuous nonnegative function $\delta: [0,2\pi] \to R$ will be chosen so as to satisfy the following requirements:

(1) $\delta(\theta) < \min\{d(P,g(P)): P = (r,\theta)$ and $1 \le r \le 3\frac{1}{20}\}$,

(2) $\delta(\theta) = 0$ if and only if $F = (r,\theta)$ for some r with $1 \le r \le 3\frac{1}{20}$, and

(3) $\delta(0) = \delta(2\pi)$.

As a matter of convenience we will do the whole construction in the Cartesian plane. Let $\Pi(x,y) = ((y + 1\frac{1}{20})\cos x, (y + 1\frac{1}{20})\sin x)$ and then let $\tilde{A} = \Pi^{-1}(A) = R \times [-\frac{1}{20}, 2]$. We will therefore construct a homeomorphism h from $\tilde{A}$ onto $\tilde{A}$ which moves points on $y = -\frac{1}{20}$ to points on $y = -\frac{1}{20}$ with greater x-coordinates and points on $y = 2$ to points on $y = 2$ with smaller x-coordinates and so that the homeomorphism g from A onto A which satisfies $g\Pi = \Pi h$ has exactly one fixed point in A, which lies in intA. Also we will construct a continuous nonnegative map $\delta: R \to R$ which satisfies:

(1') $\delta(x) < \min\{d(P,h(P)): P = (x,y)$ for some y in $[-\frac{1}{20}, 2]\}$,

(2') $\delta(x) = 0$ if and only if $\Pi(x,y) = F$ for some y in $[-\frac{1}{20}, 2]$, and

(3') $\delta(x) = \delta(x + 2n\pi)$ for all real numbers x and all integers n.

Now in the Cartesian plane the appropriate definition for δ-chain (still in Birkhoff's sense) is the following: The sequence of points P_0, P_1, . . . , P_n is a δ-chain if P_0 is in $y = -\frac{1}{20}$, for each k, $0 \le k < n$, P_k is in $\tilde{A}$ and for each k, $0 \le k < n$, $P_{k+1} = h(P_k) + (0,d_k)$ for some d_k such that $0 \le d_k < \delta(h(P_k))$. Further define the sets $\tilde{M}_n$ to be $\{P \in \tilde{A}: P = P_N$ for some δ-chain $P_0, P_1, \ldots, P_N\}$, let $\tilde{M} = \bigcup_{N=0}^{\infty} \tilde{M}_N$, and finally let $\tilde{S}$ be the complement of the closure of the unbounded component of the complement of the closure of $\tilde{M} \cup \{(x,y): y < -\frac{1}{20}\}$, minus $\{(x,y): y < -\frac{1}{20}\}$. Note that if the function δ, when restricted to $[0,2\pi]$ also satisfies (1), (2), and (3) in addition to (1'), (2'), and (3') then $\Pi(\tilde{M}_n) = M_n$, $\Pi(\tilde{M}) = M$, $\Pi(\tilde{S}) = S$, and $\Pi(\partial\tilde{S}) = \partial S$ if $\partial\tilde{S}$ is the boundary of $M \cup \{(x,y): y < -\frac{1}{20}\}$. So in the sequel we will consider δ-chains in the Cartesian plane only. For the particular homeomorphism h and the particular continuous function δ which we will construct, $\partial\tilde{S}$ is computable in an elementary fashion. We will then compute that part of $\partial\tilde{S}$ which lies in $([-\frac{\pi}{4},\frac{\pi}{4}] \cup [\frac{3\pi}{4},\frac{5\pi}{4}])$ and show that $\partial\tilde{S}$ is not a simple curve (hence ∂S is not a simple closed curve) and that, moreover, there is no simple curve in $\overline{\tilde{S}} \setminus h(\tilde{S})$ which separates $y = -\frac{1}{20}$, from $y = 2$ (hence there is no simple closed curve in $\overline{S} \setminus g(S)$ which separates $r = 1$ from $r = 3\frac{1}{2}$).

Constructing the Homeomorphism h and the Continuous Map δ

In this section we construct the homeomorphism $h: R \times [-\frac{1}{20},2] \twoheadrightarrow R \times [-\frac{1}{20},2]$ and then the continuous map $\delta: R \to R$. The homeomorphism h is a lift of a twist homeomorphism g from A onto A which has exactly one fixed point and δ is a continuous nonnegative

function which satisfies the requirements (1'), (2'), and (3') from the previous section and its restriction to $[0,2\pi]$ satisfies the requirements (1), (2), and (3) from the previous section. In these last two sections we will simplify notation by writing M_n, M, and S for $\tilde{M}_n$, $\tilde{M}$, and $\tilde{S}$, if the context is clear.

DEFINITION 3.1. Let h_1 be the homeomorphism from $R \times [-\frac{1}{20}, 2]$ onto $R \times [-\frac{1}{20}, 2]$ defined by $h_1(x,y) = (x, f_x(y))$ where each homeomorphism $f_x\colon [-\frac{1}{20}, 2] \twoheadrightarrow [-\frac{1}{20}, 2]$ is defined in the following way:

(1) if $x = 0$, $\pi/2^n$, or, $2\pi - \pi/2^n$ for $n = 2, 3, 4, \ldots$, let

$$f_x(y) = \begin{cases} y & -\frac{1}{20} \le y \le \frac{5}{10} \\ \frac{1}{4}y + \frac{3}{8} & \frac{5}{10} < y \le \frac{9}{10} \\ 4y - 3 & \frac{9}{10} < y \le 1 \\ y & 1 < y \le 2, \end{cases}$$

(2) if $x = 3\pi/2^n$ or $2\pi - 3\pi/2^n$, for $n = 4, 5, 6, \ldots$, let

$$f_x(y) = \begin{cases} y & -\frac{1}{20} \le y \le 0 \\ \frac{6}{15}y & 0 < y \le \frac{3}{2^{n+1}} \\ \frac{5\cdot 2^n - 6}{5\cdot 2^n - 15}y - \frac{9}{5\cdot 2^{n+1} - 30} & \frac{3}{2^{n+1}} < y \le \frac{1}{2} \\ \frac{1}{4}y + \frac{3}{8} & \frac{1}{2} < y \le \frac{9}{10} \\ 4y - 3 & \frac{9}{10} < y \le 1 \\ y & 1 < y \le 2, \text{ and} \end{cases}$$

(3) if $\frac{3\pi}{4} \le x \le \frac{5\pi}{4}$, let

$$f_x(y) = \begin{cases} \frac{1}{2}y - \frac{1}{40} & -\frac{1}{20} \le y \le \frac{1}{20} \\ y - \frac{1}{20} & \frac{1}{20} < y \le \frac{7}{10} \\ \frac{3}{4}y + \frac{1}{8} & \frac{7}{10} < y \le \frac{9}{10} \\ 2y - 1 & \frac{9}{10} < y \le 1 \\ y & 1 < y \le 2. \end{cases}$$

So far we have defined homeomorphisms f_x for every $x \in B$ where B is the union of $\{0, \pi/2^n, 2\pi - \pi/2^n: n = 2, 3, \ldots\}$, $\{3\pi/2^n, 2\pi - 3\pi/2^n: n = 4, 5, 6, \ldots\}$, and $\{x: \frac{3\pi}{4} \le x \le \frac{5\pi}{4}\}$. If $x \in [0,2\pi) \setminus B$, let b be the smallest element of B which is larger than x and let a be the largest element of B which is smaller than x. There is a unique number λ between 0 and 1 so that $x = \lambda b + (1 - \lambda)a$. Define f_x by $f_x(y) = \lambda f_b(y) + (1 - \lambda) f_a(y)$. If $x \notin [0,2\pi)$ then there is exactly one integer n and one $\bar{x} \in [0,2\pi)$ so that $x = \bar{x} + 2n\pi$. Define f_x by $f_x(y) = f_{\bar{x}}(y)$.

The following lemma is an immediate consequence of the definition of h_1.

LEMMA 3.2. The homeomorphism $h_1: R \times [-\frac{1}{20}, 2] \twoheadrightarrow R \times [-\frac{1}{20}, 2]$ has the following properties:

(1) $h_1: R \times [-\frac{1}{20}, 1] \twoheadrightarrow R \times [-\frac{1}{20}, 1]$,

(2) $h_1: R \times [-\frac{1}{20}, \frac{9}{10}] \to R \times [-\frac{1}{20}, \frac{9}{10}]$,

(3) $h_1(x,y) = (x,y)$ if $1 \le y \le 2$, and

(4) $h_1(x + 2n\pi, y) = h_1(x,y) + (2n\pi, 0)$ for every integer n.

DEFINITION 3.3. Let h_2 be the homeomorphism from $R \times [-\frac{1}{20}, 2]$ onto $R \times [-\frac{1}{20}, 2]$ defined by $h_2(x,y) = (k_y(x), j_x(y))$ where the homeomorphism $k_y: R \to R$ and $j_x: [-\frac{1}{20}, 2] \twoheadrightarrow [-\frac{1}{20}, 2]$ are defined in the following way. Define k_y by

$$k_y(x) = \begin{cases} x + \pi & -\frac{1}{20} \le y \le 1 \\ x + (-2\pi y + 3\pi) & 1 < y \le 2. \end{cases}$$

If $x = 0$, let $j_x(y) = y$. If $x = \pi$, let j_x be defined by

$$j_x(y) = \begin{cases} y & -\frac{1}{20} \le y \le 1 \\ (y-1)^2 + 1 & 1 < y < 2. \end{cases}$$

If $x \in (0,\pi)$, then there is a number λ between 0 and 1 so that $x = \lambda\pi + (1-\lambda)\cdot 0$; so define j_x by $j_x(y) = \lambda\, j_\pi(y) + (1-\lambda)j_0(y)$. Note that $j_x(y) = \frac{x}{\pi}\,((y-1)^2 + 1) + (1 - \frac{x}{\pi})y$. If $x \in (\pi, 2\pi)$, let $j_x(y) = j_{2\pi - x}(y)$. Finally, if $x \notin [0,2\pi)$, then there is exactly one integer n and one number $\bar{x} \in [0,2\pi)$; so that $x = \bar{x} + 2n\pi$, so define $j_x(y) = j_{\bar{x}}(y)$.

The following lemma is an immediate consequence of the definition of h_2:

LEMMA 3.4. The homeomorphism $h_2\colon R \times [-\frac{1}{20}, 2] \twoheadrightarrow R \times [-\frac{1}{20}, 2]$ has the following properties:

(1) $h_2(x,y) = (x + \pi, y)$ if $-\frac{1}{20} \le y \le 1$,

(2) $h_2(x,y) = (x - \pi, y)$ if $y = 2$,

(3) $h_2(x + 2n\pi, y) = h_2(x,y) + (2n\pi, 0)$ for every integer n, and

(4) $\{(0 + 2n\pi, \frac{3}{2})$: n is an integer$\}$ is the set of fixed points of h_2.

DEFINITION 3.5. Let $h = h_2 \cdot h_1$ and let $\hat{h}_x(y)$ be the y-coordinate of $h(x,y)$.

The proposition which follows is an immediate consequence of the definition of h, and in turn the definitions of h_1 and h_2.

PROPOSITION 3.6. The homeomorphism $h\colon R \times [-\frac{1}{20}, 2] \twoheadrightarrow R \times [-\frac{1}{20}, 2]$ has the following properties:

(1) $h(x,y) = (x + \pi, y)$ if $y = -\frac{1}{20}$,

(2) $h(x,y) = (x - \pi, y)$ if $y = 2$,

(3) $h(x,y) = (x + \pi,\ f_x(y))$ if $-\frac{1}{20} \le y \le 1$,

(4) $F = \{(0 + 2n\pi, \frac{3}{2}): n$ is an integer$\}$ is the set of fixed points of h,

(5) $h(x + 2n\pi, y) = h(x,y) + (2n\pi, 0)$ for every integer n, and

(6) $h(x,y) \neq (x,y) + (2n\pi, 0)$ unless $n = 0$ and $(x,y) \in F$.

Note here that the homeomorphism $g: A \twoheadrightarrow A$ induced by h; that is, the unique homeomorphism g which satisfies $g\Pi = \Pi h$, is a twist homeomorphism of A which has $g(1,\theta) = (1, \theta + \pi)$ and $g(3\frac{1}{20}, \theta) = g(3\frac{1}{20}, \theta - \pi)$. Furthermore, the only fixed point of g is $\Pi(F) = (0, 2\frac{11}{20})$. Before defining the function δ we prove the following lemma.

LEMMA 3.7. If $0 \le x \le \pi$, then $d((x,y), h(x,y)) \ge \frac{3x}{16\pi}$ and if $\pi < x \le 2\pi$ then $d((x,y), h(x,y)) \ge \frac{3(2\pi - x)}{16\pi}$.

PROOF. Suppose $x \in (0,\pi]$. If $-\frac{1}{20} \le y \le \frac{5}{4}$ or $\frac{7}{4} \le y \le 2$, then $d((x,y), h(x,y)) \ge |k_y(x) - x| \ge \min\{|k_y(x) - x|: -\frac{1}{20} \le y \le \frac{5}{4}$ or $\frac{7}{4} \le y \le 2\}$, but this minimum occurs at $y = \frac{5}{4}$ and $y = \frac{7}{4}$ where $|k_y(x) - x| = |-2\pi y + 3\pi| = \frac{\pi}{2}$. If $\frac{5}{4} \le y \le \frac{7}{4}$, then $d((x,y), h(x,y)) \ge |j_x(y) - y| \ge \min\{|j_x(y) - y|: \frac{5}{4} \le y \le \frac{7}{4}\}$. Since $d(j_x(y) - y)/dy = \frac{x}{\pi}(2y - 2) - \frac{x}{\pi} = 0$ at $y = \frac{3}{2}$, the minimum of $|j_x(y) - y|$ on $\frac{5}{4} \le y \le \frac{7}{4}$ occurs at either $y = \frac{5}{4}$, $y = \frac{3}{2}$, or $y = \frac{7}{4}$. Computing we get $|j_x(\frac{5}{4}) - \frac{5}{4}| = \frac{3x}{16\pi}$, $|j_x(\frac{3}{2}) - \frac{3}{2}| = \frac{x}{4\pi}$, and $|j_x(\frac{7}{4}) - \frac{7}{4}| = \frac{11x}{16\pi}$, and so the minimum value is $\frac{3x}{16\pi}$. Hence $d((x,y), h(x,y)) \ge \frac{3x}{16\pi}$ for $\frac{5}{4} \le y \le \frac{7}{4}$. Now since $\frac{3x}{16\pi} < \frac{\pi}{2}$ for all $x \in (0,\pi]$, $d((x,y), h(x,y)) \ge \frac{3x}{16\pi}$ for all $x \in (0,\pi]$. Clearly if $x = 0$, this same inequality holds. If $x \in (\pi, 2\pi)$ and $-\frac{1}{20} \le y \le \frac{5}{4}$ or $\frac{7}{4} \le y \le 2$ then, as in the first case, $d((x,y), h(x,y)) \ge \frac{\pi}{2}$. If $\frac{5}{4} \le y \le \frac{7}{4}$, then $d((x,y), h(x,y)) \ge \min\{|j_x(y) - y|: \frac{5}{4} \le y \le \frac{7}{4}\}$ and by definition $j_x(y) = j_{2\pi - x}(y)$ so that $\min|j_x(y) - y| = \min|j_{2\pi-x}(y) - y| = \frac{3(2\pi - x)}{16\pi}$. Hence for $x \in (\pi, 2\pi)$, $d((x,y), h(x,y)) \ge \frac{3(2\pi - x)}{16\pi}$. □

DEFINITION 3.8. Define the continuous function $\delta: R \to R$ first on $[0,2\pi)$ by

$$\delta(x) = \begin{cases} \dfrac{x}{10\pi} & 0 \le x \le \dfrac{\pi}{2} \\ \dfrac{1}{20} & \dfrac{\pi}{2} < x \le \dfrac{3\pi}{2} \\ \dfrac{2\pi - x}{10\pi} & \dfrac{3\pi}{2} < x < 2\pi. \end{cases}$$

If $x \notin [0,2\pi)$, then there is exactly one integer n and one number $\bar{x} \in [0,2\pi)$ so that $x = \bar{x} + 2n\pi$, define $\delta(x) = \delta(\bar{x})$.

LEMMA 3.9. For every real number x, $\delta(x) < \min\{d((x,y), h(x,y)): -\frac{1}{20} \le y \le 2\}$.

PROOF. It is sufficient to prove that the inequality holds for $x \in [0,2\pi)$, because if $x = \bar{x} + 2n\pi$ where $\bar{x} \in [0,2\pi)$ then $d((x,y), h(x,y)) = d((\bar{x},y), h(\bar{x},y))$ and $\delta(x) = \delta(\bar{x})$. So first, if $x \in [0,\frac{\pi}{2}]$, then

$$\delta(x) = \frac{x}{10\pi} < \frac{3x}{16\pi} \le \min\{d((x,y), h(x,y)): -\frac{1}{20} \le y \le 2\}.$$

If $x \in (\frac{\pi}{2},\pi]$, then

$$\delta(x) = \frac{1}{20} < \frac{3\cdot\pi/2}{16\pi} \le \frac{3x}{16\pi} \le \min\{d((x,y), h(x,y)): -\frac{1}{20} \le y \le 2\}.$$

If $x \in (\pi,\frac{3\pi}{2})$, then

$$\delta(x) = \frac{1}{20} < \frac{3(2\pi - 3\pi/2)}{16\pi} \le \frac{3(2\pi - x)}{16\pi} \le \min\{d((x,y),h(x,y)): -\frac{1}{20} \le y \le 2\}.$$

And finally, if $x \in [\frac{3\pi}{2},2\pi)$, then

$$\delta(x) = \frac{2\pi - x}{10\pi} < \frac{3(2\pi - x)}{16\pi} \le \min\{d((x,y), h(x,y)): -\frac{1}{20} \le y \le 2\}.$$

Hence for all x, $\delta(x) < \min\{d((x,y), h(x,y)): -\frac{1}{20} \le y \le 2\}$. □

PROPOSITION 3.10. The function $\delta: R \to R$ is a continuous, nonnegative function with the following additional properties:

(1) $\delta(x + 2n\pi) = \delta(x)$ for all integers n,

(2) $\delta(x) < \min\{d((x,y), h(x,y)): -\frac{1}{20} \le y \le 2\}$, and

(3) $\delta(x) = 0$ if and only if $x = 0 + 2n\pi$ for some integer n, that is $\delta(x) = 0$ if and only if $(x,y) \in \Pi(F)$ for some $y \in [-\frac{1}{20}, 2]$.

PROOF. That δ is continuous, nonnegative, and has properties (1) and (3) is immediate from the definition of δ. Property (2) is the previous lemma. $\square$

Note that if $\delta: [0,2\pi] \to R$ is defined the same as this function δ, that is by

$$\delta(\theta) = \begin{cases} \dfrac{\theta}{10\pi} & 0 \le \theta \le \dfrac{\pi}{2} \\[2ex] \dfrac{1}{20} & \dfrac{\pi}{2} < \theta \le \dfrac{3\pi}{2} \\[2ex] \dfrac{2\pi - \theta}{10\pi} & \dfrac{3\pi}{2} < \theta \le 2\pi, \end{cases}$$

then $\delta(\theta) = 0$ if and only if $\theta = 0$ and $\delta(\theta) < \min\{d((r,\theta), g(r,\theta)): 1 \le r \le 3\frac{1}{20}\}$. This last inequality can be shown by reasoning similar to that in the proof of lemma 3.7. Suppose $\theta \in [0,\pi]$ and $1 \le r \le 2\frac{6}{20}$ or $2\frac{16}{20} \le r \le 3\frac{1}{20}$. If $(r_1, \theta_1) = g(r,\theta)$ then $\frac{\pi}{2} \le |\theta_1 - \theta| \le \Pi$ so that $\min\{d((r,\theta), g(r,\theta)): 1 \le r \le 2\frac{6}{20}$ or $2\frac{16}{20} \le r \le 3\frac{1}{20}\} \ge \frac{\sqrt{2}}{2}$.

Since $\frac{\sqrt{2}}{2} > \frac{3\theta}{16\pi}$ for all $\theta \in [0,\pi]$, we have $\min d((r,\theta), g(r,\theta)):$ $1 \le r \le 2\frac{6}{20}$ or $2\frac{16}{20} \le r \le 3\frac{1}{20}\} > \frac{3\theta}{16\pi}$. Now suppose $2\frac{6}{20} \le r \le 2\frac{16}{20}$, and let $(r_1, \theta_1) = g(r,\theta)$; then $d((r,\theta), g(r,\theta)) \ge |r_1 - r| = |j_x(y) - y|$ where $x = \theta$; $(x,y) \in \Pi^{-1}(r,\theta)$, hence the same estimates apply as in lemma 3.7, so that $d((r,\theta), g(r,\theta)) \ge \frac{3\theta}{16\pi}$. As before, if $\theta \in (\pi, 2\pi)$, then

$d((r,\theta), g(r,\theta)) > \frac{3(2\pi - \theta)}{16\pi}$. Hence $\delta(\theta) < \min\{d((r,\theta), g(r,\theta)): 1 \leq r \leq 3\frac{1}{2}\}$.

Determining ∂S

Using the function δ: $R \to R$ and the homeomorphism h: $R \times [-\frac{1}{20}, 2] \twoheadrightarrow R \times [-\frac{1}{20}, 2]$ construct the sets $M_n = \{P: P = P_n$ for some δ-chain $P_0, P_1, \ldots, P_n\}$. For each positive integer n let ∂M_n denote the boundary of $M \cup \{(x,y): y \leq -\frac{1}{20}\}$.

LEMMA 3.11. For each positive integer n ∂M_n is a continuous function from R into $[-\frac{1}{20}, \frac{9}{10}]$ which is periodic in the sense that $\partial M_n(x + 2\pi) = \partial M_n(x)$ for all x.

PROOF. Since $M_1 = \{(x,y): -\frac{1}{20} \leq y < \delta(x)\}$ and $\delta(x) \leq \frac{1}{20}$ for all x, $M_1 \subset R \times [-\frac{1}{20}, \frac{9}{10}]$. Suppose $M_k \subset R \times [-\frac{1}{20}, \frac{9}{10}]$ for each integer k, $1 \leq k < n$. If $P \in M_n$, there is some point (x,y) in $h(M_{n-1})$ so that $P = (x,y) + (0,\varepsilon)$ where $0 \leq \varepsilon < \delta(x) \leq \frac{1}{20}$. Since $h(M_{n-1}) \subset h(R \times [-\frac{1}{20}, \frac{9}{10}]) \subset R \times [-\frac{1}{20}, \frac{8}{10}]$, $M_n \subset R \times [-\frac{1}{20}, \frac{8}{10} + \frac{1}{20}] \subset R \times [-\frac{1}{20}, \frac{9}{10}]$. Hence $\overline{M}_n \subseteq R \times [-\frac{1}{20}, \frac{9}{10}]$ so that $\partial M_n \subset R \times [-\frac{1}{20}, \frac{9}{10}]$, that is, the set ∂M_n is contained in $R \times [-\frac{1}{20}, \frac{9}{10}]$. Next we show, also by induction, that ∂M_n is a continuous function. [Note here that in general this is not the case.] First since $\partial M_1 = \{(x,y): y = \delta(x)\}$, we can set $y = \partial M_1(x)$ if and only if $(x,y) \in \partial M_1$. Notice that $\partial M_1(x) = \delta(x) = \text{LUB}\{y: (x,y) \in M_1\}$, and ∂M_1 is a continuous function. Suppose for each integer k, $1 \leq k < n$, ∂M_k is a continuous function and set $y = \partial M_k(x)$ if and only if $(x,y) \in \partial M_k$. In this case $\partial M_k(x) = \text{LUB}\{y: (x,y) \in M_k\}$. Since h_1 is a vertical homeomorphism, $h_1(\partial M_{n-1})$ is also a continuous function;

since h_2 is a horizontal translation of π units on $R \times [-\frac{1}{20}, 1]$, $h_2 h_1(\partial M_{n-1}) = h(\partial M_{n-1})$ is a continuous function. Let $\Delta(h(\partial M_{n-1})) = \{(x, y + \delta(x)): (x,y) \in h(\partial M_{n-1})\}$. Then $\Delta(h(\partial M_{n-1}))$ is a continuous function and $\Delta \cdot h \cdot \partial M_{n-1}(x) = \hat{h}_{x-\pi}(\partial M_{n-1}(x - \pi)) + \delta(x)$ where $\hat{h}_x(y)$ is the y-coordinate of $h(x,y)$. It remains to show that $\Delta(h(\partial M_{n-1})) = \partial M_n$. Let $\psi(x) = \text{LUB}\ \{y: (x,y) \in M_n\}$. Therefore

$\psi(x) = \text{LUB}\ \{y + \varepsilon: (x,y) \in h(M_{n-1})$ and $0 \le \varepsilon < \delta(x)\}$. Now $\partial M_{n-1}(x - \pi) = \text{LUB}\ \{y: (x - \pi, y) \in \partial M_{n-1}\}$. Since $\hat{h}_x(\partial M_{n-1}(x - \pi)) = \text{LUB}\ \{y: (x,y) \in h(\partial M_{n-1})\}$, so that $\hat{h}_x(\partial M_{n-1}(x - \pi)) + \delta(x) = \text{LUB}\ \{y + \varepsilon: (x,y) \in h(\partial M_{n-1})$ and $0 \le \varepsilon < \delta(x)\}$ so that

$\psi(x) = \hat{h}_x(\partial M_{n-1}(x - \pi)) + \delta(x)$, ψ is a continuous function. Since $\psi = \{(x, \psi(x)): x \in R\} \subset \partial M_n$ and ψ is connected, $\psi = \partial M_n$ so that ∂M_n is a continuous function and $\partial M_n(x) = \hat{h}_{x-\pi}(\partial M_{n-1}(x - \pi)) + \delta(x)$. □

LEMMA 3.12. Let $L(x) = \text{LUB}\ \{y: (x,y) \in M\}$. For each $x \in R$ the following hold:

(1) $\partial M_n(x) \le \partial M_{n+1}(x) \le L(x) < 1$ for every nonnegative integer n,

(2) $L(x) = \lim\limits_{n\to\infty} \partial M_n(x)$,

(3) $L(x) = \lim\limits_{n\to\infty} \partial M_{k+2n}(x)$, for every nonnegative integer k, and

(4) $L(x) = L(x + 2k\pi)$ for every integer k.

PROOF. Since $M_n \subseteq M_{n+1} \subseteq M \subseteq R \times [-\frac{1}{20}, \frac{9}{10}]$ for every nonnegative integer n, $\partial M_n(x) \le \partial M_{n+1}(x) \le L(x) < 1$. Since $\{\partial M_n(x)\}_{n=0}^{\infty}$ is non-decreasing and bounded for each x, $\lim\limits_{n\to\infty} \partial M_n(x)$ exists and since $M = \bigcup\limits_{n=0}^{\infty} M_n$ $L(x) = \lim\limits_{n\to\infty} \partial M_n(x)$. Because any subsequence of a convergent sequence converges to the same limit, $L(x) = \lim\limits_{n\to\infty} \partial M_{k+2n}(x)$ for every nonnegative integer k. For each nonnegative integer n, and each integer k, $(x,y) \in M_n$ if and only if $(x + 2k\pi, y) \in M_n$; therefore $(x,y) \in M$ if and only if $(x + 2k\pi, y) \in M$; hence $L(x) = L(x + 2k\pi)$ for all integers k. □

LEMMA 3.13. The set $M \cup \{(x,y): y \le -\frac{1}{20}\}$ is simply connected.

PROOF. Let $A_n = (M_n \cup \{(x,y): y \le -\frac{1}{20}\})^c$, then $A_n = \{(x,y): y \ge \partial M_n(x)\}$. Each A_n is closed, connected, and contains $\{(x,y): y \ge 2\}$, and for every nonnegative integer n, $A_{n+1} \subseteq A_n$ so that $\bigcap_{n=0}^{\infty} A_n$ is connected. Adjoining the point at infinity to $\bigcap_{n=0}^{\infty} A_n$ results in a connected set E (now in S^2) and $E = (M \cup \{(x,y): y \le -\frac{1}{20}\})^c$ (in S^2) so that $M \cup \{(x,y): y \le -\frac{1}{20}\}$ must be simply connected. (An open connected set in E^2 is simply connected if its complement in S^2 is connected.) □

Note that this last lemma does not quite show that M = S. Before proving the next lemma, which is at the heart of the calculation, we need a definition.

DEFINITION 3.14. For each $(x,y) \in R \times [-\frac{1}{20}, 2]$ let $\psi_x(y) = f_{x+\pi}(f_x(y) + \delta(x+\pi)) + \delta(x)$.

LEMMA 3.15. For every $x \in R$ and every nonnegative integer k $L(x) = \lim_{n\to\infty} \psi_x^n(-\frac{1}{20}) = \lim_{n\to\infty} \psi_x^n(\partial M_k(x))$.

PROOF. Note that $\partial M_0(x) = -\frac{1}{20} = \psi_x^0(x)$. Suppose that for each integer k, $1 \le k \le n$, $\partial M_{2k}(x) = \psi_x^k(-\frac{1}{20})$. Then $\partial M_{2n+1}(x) = \hat{h}_{x-\pi}(\partial M_{2n}(x-\pi)) + \delta(x)$, by the last sentence in the proof of lemma 3.11. So $\partial M_{2n+1}(x) = f_{x-\pi}(\partial M_{2n}(x-\pi)) + \delta(x)$, since $\partial M_{2n}(x-\pi)$ is in $R \times [-\frac{1}{20}, 1]$, and on $R \times [-\frac{1}{20}, 1]$ h_2 is a horizontal translation so that the vertical motion is all due to h_1. So

$$\partial M_{2n+2}(x) = \hat{h}_{x-\pi}(\partial M_{2n+1}(x-\pi)) + \delta(x)$$

$$= \hat{h}_{x-\pi}(f_{x-2\pi}(\partial M_{2n}(x-2\pi)) + \delta(x-\pi)) + \delta(x)$$

$$= f_{x-\pi}(f_{x-2\pi}(\partial M_{2n}(x-2\pi) + \delta(x-\pi)) + \delta(x).$$

Now each of h, ∂M_{2n}, and δ is periodic so that x may be replaced by $x + 2\pi$ in the last equation so that $\partial M_{2n+2}(x) = f_{x-\pi}\left(f_x(\partial M_{2n}(x)) + \delta(x - \pi)\right) + \delta(x)$. So using the induction hypothesis, we have $\partial M_{2n+2}(x) = f_{x-\pi}\left(f_x(\psi_x^n(-\frac{1}{20})) + \delta(x + \pi)\right) + \delta(x)$; hence $\partial M_{2n+2}(x) = \psi_x^{n+1}(-\frac{1}{20})$. Hence $\lim\limits_{n\to\infty} \psi_x^n(-\frac{1}{20}) = \lim\limits_{n\to\infty} \partial M_{2n}(x) = L(x)$ and $\lim\limits_{n\to\infty} \psi_x^n(\partial M_k(x)) = \lim\limits_{n\to\infty} \partial M_{k+2n}(x) = L(x)$. □

LEMMA 3.17. If $x \in [2k\pi - \frac{\pi}{4}, 2k\pi + \frac{\pi}{4}]$ for some integer k then

$$\psi_x(y) = \begin{cases} \frac{1}{2} f_x(y) + \delta(x) & -\frac{1}{20} \le y \le 0 \\ f_x(y) + \delta(x) & 0 < y \le \frac{9}{10}. \end{cases}$$

PROOF. From the previous lemma we have $\psi_x(y) = f_{x+\pi}(f_x(y) + \delta(x + \pi)) + \delta(x)$. Now $\delta(x + \pi) = 20$ for all $x \in [2k\pi - \frac{\pi}{4}, 2k\pi + \frac{\pi}{4}]$. If $-\frac{1}{20} \le y \le 0$, then $f_x(y) = y$ so that $-\frac{1}{20} \le f_x(y) \le 0$, and then $0 \le f_x(y) + \frac{1}{20} \le \frac{1}{20}$; therefore $f_{x+\pi}(f_x(y) + \delta(x)) = \frac{1}{2}(f_x(y) + \frac{1}{20}) - \frac{1}{40} = \frac{1}{2} f_x(y)$; and thus $\psi_x(y) = \frac{1}{2} f_x(y) + \delta(x)$. If $0 < y \le \frac{9}{10}$, then $0 < f_x(y) \le \frac{6}{10}$ so that $\frac{1}{20} < f_x(y) + \frac{1}{20} \le \frac{13}{20}$; therefore $f_{x+\pi}(f_x(y) + \frac{1}{20}) = (f_x(y) + \frac{1}{20}) - \frac{1}{20}$, and thus $\psi_x(y) = f_x(y) + \delta(x)$. □

LEMMA 3.18. If $x \in [2k\pi + \frac{2\pi}{4}, 2k\pi + \frac{5\pi}{4}]$ for some integer k, then the following hold:

(1) there is a smallest integer $N = N(x)$ so that $0 \le \psi_{x+\pi}^N(-\frac{1}{20}) < \delta(x + \pi)$, and

(2) if $n > N$ then $\partial M_{2n-1}(x) = (\psi_{x+\pi})^n(-\frac{1}{20}) + \frac{1}{20} - \delta(x + \pi)$.

PROOF. Without loss of generality, assume $x \in [\frac{3\pi}{4}, \frac{5\pi}{4}]$. If $y \le 0$, then $f_{x+\pi}(y) = y$ and $f_{x+\pi}(y) + \delta(x) = y + \frac{1}{20} \le \frac{1}{20}$; so that $f_x(y + \frac{1}{20}) = \frac{1}{2}(y + \frac{1}{20}) - \frac{1}{40}$. Hence $\psi_{x+\pi}(y) = f_x(f_{x+\pi}(y) + \delta(x)) + \delta(x + \pi) =$

$\frac{1}{2} y + \left|\frac{x - \pi}{10\pi}\right|$. Now $\left|\frac{x - \pi}{10\pi}\right| \le \frac{\pi/4}{10\pi} = \frac{1}{40}$, hence as long as $\frac{1}{2} y + \left|\frac{x - \pi}{10\pi}\right|$ remains less than zero we can iterate $\psi_{x+\pi}$ using the formula $\psi_{x+\pi}(y) = \frac{1}{2} y + \left|\frac{x - \pi}{10\pi}\right|$. Now $(\psi_{x+\pi})^n(-\frac{1}{20}) = \frac{1}{2^n} (-\frac{1}{20}) + \left(1 + \frac{1}{2} + \ldots + \frac{1}{2^{n-1}}\right)\left|\frac{x - \pi}{10\pi}\right|$ so that $(\psi_{x+\pi})^n = \frac{1}{2^n} (-\frac{1}{20}) + \left(2 - \frac{1}{2^{n-1}}\right)\left|\frac{x - \pi}{10\pi}\right|$. So, clearly there is a first N, depending on x, so that $(\psi_{x+\pi})^N(-\frac{1}{20}) \ge 0$. Therefore $(\psi_{x+\pi})^{N-1}(-\frac{1}{20}) < 0$ so that $\psi_{x+\pi}(\psi_{x+\pi})^{N-1}(-\frac{1}{20}) = \frac{1}{2} (\psi_{x+\pi})^{N-1}(-\frac{1}{20}) + \left|\frac{x - \pi}{10\pi}\right| < 0 + \delta(x + \pi)$. Hence, so far we have shown $0 \le (\psi_{x+\pi})^N(-\frac{1}{20}) < \delta(x + \pi)$. Also note that $(\psi_{x+\pi})^n(-\frac{1}{20}) \ge 0$ for all $n \ge N$. Suppose that $n \ge N + 1 = N(x) + 1$. Since $\partial M_{2n-1}(x) = \hat{h}_{x+\pi}(\partial M_{2n-2}(x + \pi)) + \delta(x) = f_{x+\pi}(\partial M_{2n-2}(x + \pi)) + \frac{1}{20} = f_{x+\pi}((\psi_{x+\pi})^{n-1}(-\frac{1}{20})) + \frac{1}{20}$, we have $\partial M_{2n-1}(x) \ge \frac{1}{20}$. Since $f_{x+\pi}: [-\frac{1}{20}, \frac{9}{20}] \twoheadrightarrow [-\frac{1}{20}, \frac{4}{10}]$ and $f_x: [-\frac{1}{20}, \frac{9}{10}] \twoheadrightarrow [-\frac{1}{20}, \frac{8}{10}]$, $\partial M_{2n-1}(x) \le \frac{4}{10} + \frac{1}{20} < \frac{7}{10}$; so that $\frac{1}{20} \le \partial M_{2n-1}(x) < \frac{7}{10}$. Therefore $\partial M_{2n}(x + \pi) = \hat{h}_x(\partial M_{2n-1}(x)) + \delta(x + \pi) = f_x(\partial M_{2n-1}(x)) + \delta(x + \pi) = \partial M_{2n-1}(x) - \frac{1}{20} + \delta(x + \pi)$. Thus we have $\partial M_{2n}(x + \pi) = \partial M_{2n-1}(x) - \frac{1}{20} + \delta(x + \pi)$ and $\partial M_{2n-1}(x) = \partial M_{2n}(x + \pi) + \frac{1}{20} - \delta(x + \pi)$ so that $\partial M_{2n-1}(x) = (\psi_{x+\pi})^n(-\frac{1}{20}) + \frac{1}{20} - \delta(x + \pi)$. □

LEMMA 3.19. If $x \in [(2k + 1)\pi - \frac{\pi}{4}, (2k + 1)\pi + \frac{\pi}{4}]$ for some integer k, then $L(x) = L(x + \pi) + \frac{1}{20} - \delta(x + \pi)$.

PROOF. By the previous lemma $\partial M_{2n-1}(x) = (\psi_{x+\pi})^n(-\frac{1}{20}) + \frac{1}{20} - \delta(x + \pi)$. Hence $L(x) = \lim_{n\to\infty} \partial M_{2n-1}(x) = \lim_{n\to\infty} \left((\psi_{x+\pi})^n(-\frac{1}{20}) + \frac{1}{20} - \delta(x + \pi)\right)$. So $L(x) = \lim_{n\to\infty} (\psi_{x+\pi})^n(-\frac{1}{20}) + \frac{1}{20} - \delta(x + \pi)$; therefore $L(x) = L(x + \pi) + \frac{1}{20} - \delta(x + \pi)$. □

Although the next proposition is well known a proof is given for the sake of completeness.

PROPOSITION 3.20. Let τ be a homeomorphism of the interval $[a,b]$ onto the interval $[a',b']$ with $a \le a' < b' \le b$, which has $\tau(a) = a'$ and $\tau(b) = b'$. If $\tau^n(x) \in [a,b]$ and $\tau^{n+1}(x) > \tau^n(x)$ for all nonnegative integers n, then $\lim_{n\to\infty} \tau^n(x) = x_0$ is the first number greater than x so that $x_0 = \tau(x_0)$.

PROOF. Since $\{\tau^n(x)\}_{n=1}^{\infty}$ is bounded and increasing $\lim_{n\to\infty} \tau^n(x)$ exists; let $x_0 = \lim_{n\to\infty} \tau^n(x)$. Now $\tau(x_0) = \tau \lim_{n\to\infty} \tau^n(x) = \lim_{n\to\infty} \tau^{n+1}(x) = x_0$. The number x_0 is the first number greater than x for which $\tau(x_0) = x_0$ holds since $[x,x_0) = \bigcup_{n=0}^{\infty} \tau^n\big([x,\tau(x))\big)$ is a disjoint union. $\square$

Now we are ready to calculate L(x) on $[0, \frac{\pi}{4}]$.

THEOREM 3.21. If $x \in [0, \frac{\pi}{4}]$ and n is an integer, $n \ge 2$ then

$$L(x) = \begin{cases} \dfrac{1}{3\cdot 2^{n+3}} \dfrac{x}{\pi/2^n - x} & \dfrac{3\pi}{2^{n+2}} \le x \le \dfrac{9\pi}{5\cdot 2^{n+1}} \\[2ex] \dfrac{2x}{15\pi} + \dfrac{1}{2} & \dfrac{9\pi}{5\cdot 2^{n+1}} < x < \dfrac{9\pi}{2^{n+3}} \\[2ex] \dfrac{1}{3\cdot 2^{n+2}} \dfrac{x}{x - \pi/2^n} & \dfrac{9\pi}{2^{n+3}} \le x \le \dfrac{3\pi}{2^{n+1}} \\[2ex] 0 & x = 0. \end{cases}$$

PROOF. The calculation proceeds as follows. Since $L(x) = \lim_{n\to\infty} \psi_x^n(-\frac{1}{20})$ for each x in $(0, \frac{\pi}{4}]$, by the preceding lemma L(x) is the first number y so that $\psi_x(y) = y$. (Note that each ψ_x is a homeomorphism of $[-\frac{1}{20}, \frac{9}{10}]$ into some subinterval of itself.) There are four cases to consider: Case (1) in which $x = \pi/2^n$, $n = 2, 3, 4, \ldots$, or $x = 0$; case (2) in which $x = 3\pi/2^2$, $n = 4, 5, \ldots$; case (3) in which $3\pi/2^{n+2} < x < \pi/2^n$, $n = 2, 3, \ldots$; and case (4) in which $\pi/2^{n+1} < x < 3\pi/2^{n+2}$. In each case a smallest solution is found

for $\psi_x(y) = y$. Since the graph of ψ_x does not cross the identity on $[-\frac{1}{20}, 0)$, it is sufficient to solve for $\psi_x(y) = y$, $y \in [0, \frac{9}{10}]$.

Case (1). Let $x = \pi/2^n$, $n = 2, 3, 4, \ldots$. Then using lemma 3.17,

$$\psi_x(y) = \begin{cases} y + \dfrac{1}{2^{n+1}\cdot 5} & 0 \le y \le \dfrac{5}{10} \\[2ex] \dfrac{1}{4} y + \dfrac{3}{8} + \dfrac{1}{2^{n+1}\cdot 5} & \dfrac{5}{10} < y \le \dfrac{9}{10} \\[2ex] 4y - 3 + \dfrac{1}{2^{n+1}\cdot 5} & \dfrac{9}{10} < y \le 1. \end{cases}$$

On $[0, \frac{5}{10}]$ the graph of ψ_x has slope 1 thus does not cross the identity. The graph of ψ_x must cross the identity on $[\frac{5}{10}, \frac{9}{10}]$ since $\psi_x(\frac{5}{10}) = \frac{5}{10} + \frac{1}{2^{n+1}\cdot 5} > \frac{5}{10}$ and $\psi_x(\frac{9}{10}) = \frac{6}{10} + \frac{1}{2^{n+1}\cdot 5} < \frac{9}{10}$. So we solve for y in $y = \frac{1}{4} y + \frac{3}{8} + \frac{1}{2^{n+1}\cdot 5}$, obtaining $y = \frac{1}{2} + \frac{1}{3\cdot 5\cdot 2^{n-1}}$. Hence at $x = \pi/2^n$, $L(x) = \frac{1}{2} + \frac{1}{3\cdot 5\cdot 2^{n-1}}$. Note that $L(\pi/2^n) > L(\pi/2^{n+1})$ and that $\lim_{n\to\infty} L(\pi/2^n) = \frac{1}{2}$. Let $x = 0$. Then using lemma 3.17

$$\psi_x(y) = \begin{cases} y + 0 & 0 \le y \le \dfrac{5}{10} \\[2ex] \dfrac{1}{4} y + \dfrac{3}{8} + 0 & \dfrac{5}{10} < y \le \dfrac{9}{10} \\[2ex] 4y - 3 + 0 & \dfrac{9}{10} < y \le 1. \end{cases}$$

Hence the first solution of $\psi_x(y) = y$ is at $y = 0$. So $L(x) = 0$.

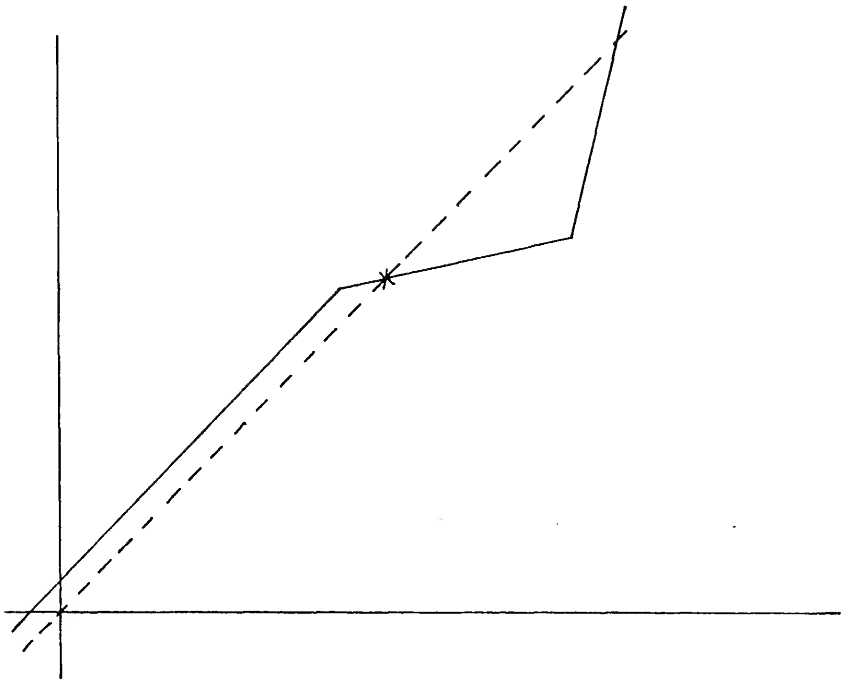

Figure 25. The graphs of ψ_x and the identity for $x = \pi/2^n$.

Case (2). Let $x = 3\pi/2^n$, $n = 4, 5, \ldots$. Then using lemma 3.17,

$$\psi_x(y) = \begin{cases} \dfrac{6}{15}\, y + \dfrac{3}{5\cdot 2^{n+1}} & 0 \le y \le \dfrac{3}{2^{n+1}} \\[2ex] \dfrac{5\cdot 2^n - 6}{5\cdot 2^n - 15}\, y - \dfrac{9}{5\cdot 2^{n+1} - 30} + \dfrac{3}{5\cdot 2^{n+1}} & \dfrac{3}{2^{n+1}} < y \le \dfrac{1}{2} \\[2ex] \dfrac{1}{4}\, y + \dfrac{3}{8} + \dfrac{3}{5\cdot 2^{n+1}} & \dfrac{1}{2} < y \le \dfrac{9}{10} \\[2ex] 4y - 3 + \dfrac{3}{5\cdot 2^{n+1}} & \dfrac{9}{10} < y \le 1. \end{cases}$$

Since $\psi_x(0) = \frac{3}{5\cdot 2^{n+1}} > 0$ and $\psi_x\left(\frac{3}{2^{n+1}}\right) = \frac{9}{5\cdot 2^{n+1}} < \frac{3}{2^{n+1}}$, the graph of ψ_x intersects that of the identity on $[0, \frac{3}{2^{n+1}}]$. So solving for y in $y = \frac{6}{15}y + \frac{3}{5\cdot 2^{n+1}}$ we obtain $y = \frac{1}{2^{n+1}}$. Hence at $x = 3\pi/2^n$, $L(x) = \frac{1}{2^{n+1}}$. Note that $\lim_{n\to\infty} L(3\pi/2^n) = 0$.

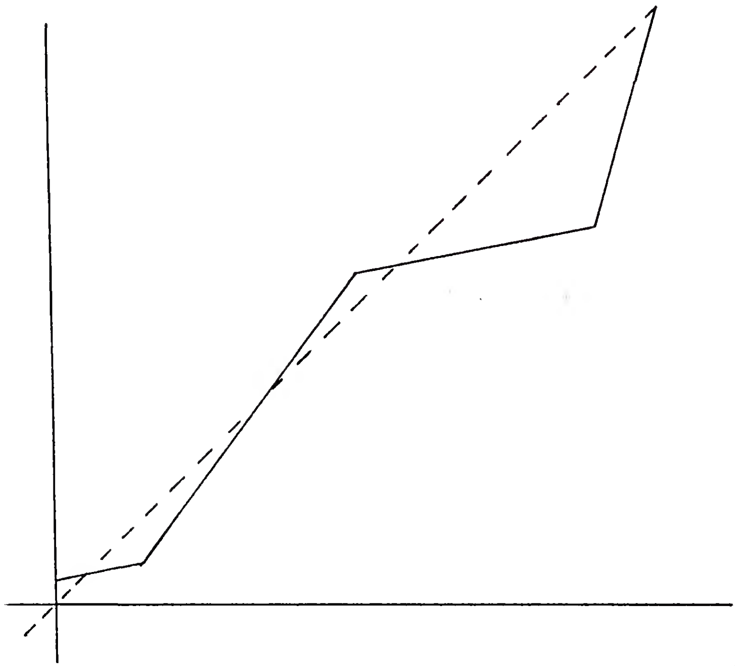

Figure 26. The graphs of ψ_x and the identity for $x = 3\pi/2^n$.

Case (3). Let $x = \lambda \frac{\pi}{2} + (1 - \lambda) \frac{3\pi}{2^{n+1}}$ for $n = 2, 3, \ldots$ and $0 < \lambda < 1$. Since δ is linear on $[0, \frac{\pi}{4}]$, we have $\delta(x) = \lambda\cdot\delta\left(\frac{\pi}{2^n}\right) + (1 - \lambda)\ \delta\left(\frac{3\pi}{2^{n+2}}\right) = \frac{3 + \lambda}{5\cdot 2^{n+3}}$. So that using lemma 3.17,

$$\psi_x(y) = \begin{cases} \lambda y + (1-\lambda)\, \frac{6}{15}\, y + \frac{3+\lambda}{5\cdot 2^{n+3}} & 0 < y < \frac{3}{2^{n+3}} \\ \lambda y + (1-\lambda)\left(\frac{5\cdot 2^n - 6}{5\cdot 2^n - 15}\, y - \frac{9}{5\cdot 2^{n+1} - 30}\right) + \frac{3+\lambda}{5\cdot 2^{n+3}} & \frac{3}{2^{n+3}} < y \le \frac{1}{2} \\ \frac{1}{4}\, y + \frac{3}{8} + \frac{3+\lambda}{5\cdot 2^{n+1}} & \frac{1}{2} < y \le \frac{9}{10} \\ 4y - 3 + \frac{3+\lambda}{5\cdot 2^{n+1}} & \frac{9}{10} < y \le 1. \end{cases}$$

There are two possibilities: Either the first segment of the graph of ψ_x crosses the identity and that intersection gives $L(x)$, or if the first segment does not intersect the identity, then the second also does not since its slope is greater than one, and then the third segment must cross the identity since $\psi_x(\frac{9}{10}) = \frac{6}{10} + \frac{3+\lambda}{5\cdot 2^{n+1}} < \frac{9}{10}$, so that intersection must be $L(x)$. These two possibilities are illustrated in Figure 27. Suppose $\psi_x(y) = y$ for some y in $[0, \frac{3}{2^{n+3}}]$. Then $y = \lambda y + (1-\lambda)\, \frac{6}{15}\, y + \frac{3+\lambda}{5\cdot 2^{n+3}}$ and so solving for y we find that $y = \frac{1}{3\cdot 2^{n+3}}\, \frac{3+\lambda}{1-\lambda}$. This will be the intersection of ψ_x with the identity only if $0 \le y \le \frac{3}{2^{n+3}}$, that is only if $0 \le \frac{1}{3\cdot 2^{n+3}}\, \frac{3+\lambda}{1-\lambda} \le \frac{3}{2^{n+3}}$. Solving for λ we find that $0 \le \lambda \le \frac{3}{5}$. Otherwise, that is for $\frac{3}{5} < \lambda \le 1$, the first solution of $\psi_x(y) = y$ is from $y = \frac{1}{4}\, y + \frac{3}{8} + \frac{3+\lambda}{5\cdot 2^{n+3}}$ so that $y = \frac{1}{2} + \frac{3+\lambda}{5\cdot 2^{n+3}}$. Now since $\lambda = \left(x - \frac{3\pi}{2^{n+1}}\right) / \left(\frac{\pi}{2^n} - \frac{3\pi}{2^{n+3}}\right)$; we have the following:

$$L(x) = \begin{cases} \frac{1}{3\cdot 2^{n+3}}\, \frac{x}{\pi/2^n - x} & \frac{3\pi}{2^{n+2}} \le x \le \frac{9\pi}{5\cdot 2^{n+1}} \\ \frac{2x}{15\pi} + \frac{1}{2} & \frac{9\pi}{5\cdot 2^{n+1}} < x \le \frac{\pi}{2^n}\ . \end{cases}$$

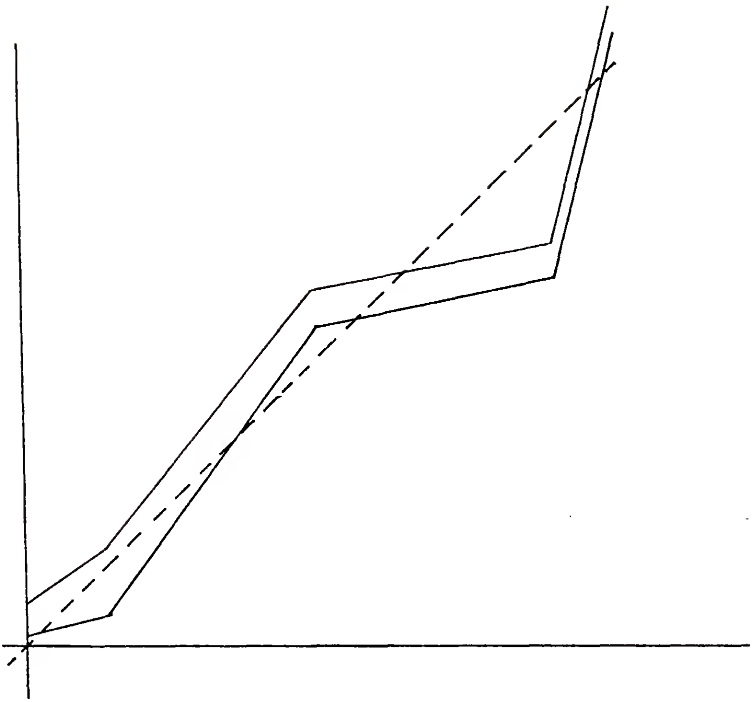

Figure 27. The graphs of two possible ψ_x and the identity for $3\pi/2^{n+2} < x < \pi/2^n$.

So the part of L which lies in $3\pi/2^{n+2} \le x \le 9\pi/5\cdot 2^{n+1}$ is a closed segment of a hyperbola with asymptotes $x = \pi/2^n$ and $y = -1/3\cdot 2^{n+3}$. The part of L which lies in $9\pi/5\cdot 2^{n+2} < x \le \pi/2^n$ is a half-open segment of the graph of $y = \frac{2x}{15\pi} + \frac{1}{2}$.

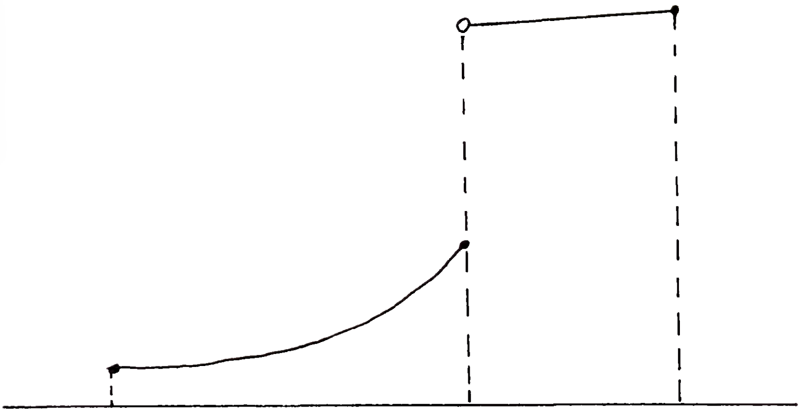

Figure 28. The graph of L for $3\pi/2^{n+2} \le x \le \pi/2^n$.

Case (4). Let $\pi/2^{n+1} \le x \le 3\pi/2^{n+2}$ for $n \ge 2$. There is some λ, $0 \le \lambda \le 1$ so that $x = \lambda \frac{3\pi}{2^{n+2}} + (1 - \lambda) \frac{\pi}{2^{n+1}}$. Since δ is linear on $[0, \frac{\pi}{4}]$, $\delta(x) = \lambda\, \delta\left(\frac{3\pi}{2^{n+2}}\right) + (1 - \lambda)\, \delta\left(\frac{\pi}{2^{n+2}}\right) = \frac{2 + \lambda}{5\cdot 2^{n+3}}$. Then, using lemma 3.17,

$$\psi_x(y) = \begin{cases} \lambda \frac{6}{15} y + (1 - \lambda)y + \frac{2 + \lambda}{5\cdot 2^{n+3}} & 0 \le y \le \frac{3}{5\cdot 2^{n+3}} \\ \lambda \left(\frac{5\cdot 2^n - 6}{5\cdot 2^n - 15} y - \frac{9}{5\cdot 2^{n+1} - 30}\right) + (1 - \lambda)y + \frac{2 + \lambda}{5\cdot 2^{n+3}} & \frac{3}{5\cdot 2^{n+3}} < y \le \frac{1}{2} \\ \frac{1}{4} y + \frac{3}{8} + \frac{2 + \lambda}{5\cdot 2^{n+3}} & \frac{1}{2} < y \le \frac{9}{10} \\ 4y - 3 + \frac{2 + \lambda}{5\cdot 2^{n+3}} & \frac{9}{10} < y \le 1. \end{cases}$$

By the same analysis as in case (3), either the first or the third segment of the graph of ψ_x gives the desired intersection. So suppose $\psi_x(y) = y$ for some y in $[0, \frac{3}{2^{n+3}}]$, that is $y = \lambda \frac{6}{15} y + (1-\lambda)y + \frac{2+\lambda}{5\cdot 2^{n+3}}$. Solving for y we obtain $y = \frac{1}{3\cdot 2^{n+3}} \frac{2+\lambda}{\lambda}$. This is the desired intersection only if $0 \leq \frac{1}{3\cdot 2^{n+3}} \frac{2+\lambda}{\lambda} \leq \frac{3}{5\cdot 2^{n+3}}$; solving for λ we obtain $\frac{1}{4} \leq \lambda \leq 1$. Hence $L(x) = \frac{1}{3\cdot 2^{n+3}} \frac{2+\lambda}{\lambda}$ for $\frac{1}{4} \leq \lambda \leq 1$. Otherwise, that is for $0 \leq \lambda < \frac{1}{4}$ the intersection is found by solving for y in $y = \frac{1}{4} y + \frac{3}{8} + \frac{2+\lambda}{5\cdot 2^{n+3}}$ to obtain $y = \frac{1}{2} + \frac{2+\lambda}{3\cdot 5\cdot 2^{n+1}}$. So $L(x) = \frac{1}{2} + \frac{2+\lambda}{3\cdot 5\cdot 2^{n+1}}$ for $0 \leq \lambda < \frac{1}{4}$. Since $\lambda = \frac{x - \pi/2^{n+1}}{3\pi/2^{n+2} - \pi/2^{n+1}}$, we have the following

$$L(x) = \begin{cases} \dfrac{2x}{15\pi} + \dfrac{1}{2} & \dfrac{\pi}{2^{n+1}} \leq x < \dfrac{9\pi}{2^{n+4}} \\[2ex] \dfrac{1}{3\cdot 2^{n+3}} \dfrac{x}{x - \dfrac{\pi}{2^{n+1}}} & \dfrac{9\pi}{2^{n+4}} \leq x \leq \dfrac{3\pi}{2^{n+2}}. \end{cases}$$

Note that the part of the graph of L in $\frac{\pi}{2^{n+1}} \leq x < \frac{9\pi}{2^{n+4}}$ is a half-open segment of the straight line $y = \frac{2x}{15\pi} + \frac{1}{2}$, and this is the same straight line that the graph of L in $\frac{9\pi}{5\cdot 2^{n+2}} < x \leq \frac{\pi}{2^{n+1}}$ is a segment of. The part of the graph of L which lies in $\frac{9\pi}{2^{n+4}} \leq x \leq \frac{3\pi}{2^{n+2}}$ is a closed segment of a hyperbola with asymptotes $x = \frac{\pi}{2^{n+1}}$ and $y = \frac{1}{3\cdot 2^{n+3}}$. Now putting the information together from the last two cases (the first two are just special instances of these) we get the required formula for L. □

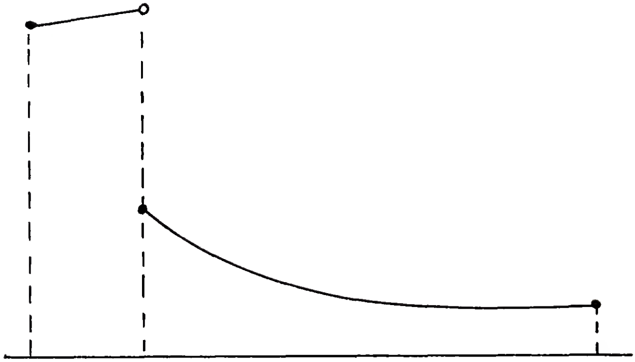

Figure 29. The graph of L on $\frac{\pi}{2^{n+1}} \leq x \leq \frac{3\pi}{2^{n+2}}$.

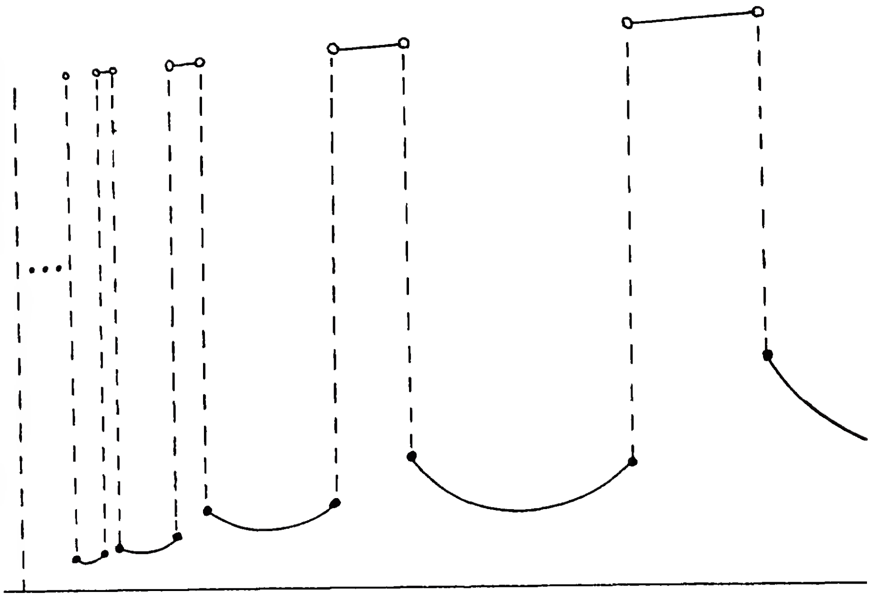

Figure 30. The graph of L on $(0, \frac{\pi}{4}]$.

Note that for $(x,y) \in [0, \frac{\pi}{4}]$ $\delta(-x) = \delta(x)$, $\delta(-x + \pi) = \delta(x + \pi)$, $f_{-x}(y) = f_x(y)$, and $f_{-x+\pi}(y) = f_{x+\pi}(y)$ so that $\psi_{-x}(y) = \psi_x(y)$. Hence $L(-x) = L(x)$. So that so far we have computed L on $[-\frac{\pi}{4}, \frac{\pi}{4}]$. Since $L(x + 2k\pi) = L(x)$ for all integers k, we have computed L on $\bigcup_{k=-\infty}^{+\infty} [2k\pi - \frac{\pi}{4}, 2k\pi + \frac{\pi}{4}]$.

COROLLARY 3.22. If $x \in [\pi, \frac{5\pi}{4}]$ then

$$L(x) = \begin{cases} \dfrac{1}{3\cdot 2^{n+3}} \dfrac{x - \pi}{\frac{\pi}{2^n} + \pi - x} + \dfrac{1}{20} - \dfrac{x - \pi}{10\pi} & \dfrac{3\pi}{2^{n+1}} + \pi \le x \le \dfrac{9\pi}{5\cdot 2^{n+1}} + \pi \\[2ex] \dfrac{2(x - \pi)}{15\pi} + \dfrac{1}{2} + \dfrac{1}{20} - \dfrac{x - \pi}{10\pi} & \dfrac{9\pi}{5\cdot 2^{n+1}} + \pi < x < \dfrac{9\pi}{2^{n+3}} + \pi \\[2ex] \dfrac{1}{3\cdot 2^{n+2}} \dfrac{x - \pi}{x - \pi - \frac{\pi}{2^{n+1}}} + \dfrac{1}{20} - \dfrac{x - \pi}{10\pi} & \dfrac{9\pi}{2^{n+3}} + \pi \le x \le \dfrac{3\pi}{2^{n+3}} + \pi \\[2ex] 0 & x = \pi. \end{cases}$$

PROOF. By lemma 3.18 $L(x) = L(x + \pi) + \frac{1}{20} - \delta(x + \pi) = L(x - \pi) + \frac{1}{20} - \delta(x - \pi)$, since $L(x - 2\pi) = L(x)$ and $\delta(x - 2\pi) = \delta(x)$ for all $x \in R$. Using the formula for L on $[0, \frac{\pi}{4}]$ given in the previous theorem we obtain the formula for L on $[\pi, \frac{5\pi}{4}]$. □

For $x \in [\frac{3\pi}{4}, \pi)$, $L(x) = L(2\pi - x)$ and since $L(x + 2k\pi) = L(x)$ for all integers k, corollary 3.22 determines the function L on $\bigcup_{k=-\infty}^{+\infty} [2k\pi + \frac{3\pi}{4}, 2k\pi + \frac{5\pi}{4}]$.

THEOREM 3.23. That part of ∂S which lies in $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$ is homeomorphic to the $\sin\frac{1}{x}$ continuum, that is to $\{(x,y): 0 < |x| \le 1$ and $y = \sin\frac{1}{x}\} \cup (\{0\} \times [-1,1])$.

PROOF. It is clear from the formula for L on $[-\frac{\pi}{4}, \frac{\pi}{4}]$ that L is continuous everywhere except on $\{0, \pm \frac{9\pi}{5\cdot 2^{n+1}}, \pm \frac{9\pi}{2^{n+4}}: n \ge 2\}$. Now

$M = \{(x,y): y < L(x)\}$ so that ∂M on $[-\frac{\pi}{4}, \frac{\pi}{4}]$ consists of the graph of L together with the union of the segments in $\{\{a\} \times [\liminf_{x\to a} L(x), \limsup_{x\to a} L(x)]$: L is not continuous at a, $-\frac{\pi}{4} < a < \frac{\pi}{4}\}$. The set S is defined to be the complement of the closure of the unbounded component of the complement of the closure of $M \cup \{(x,y): y \leq -\frac{1}{20}\}$, minus $\{(x,y): y < -\frac{1}{20}\}$. So $S \cap \{(x,y): -\frac{\pi}{4} \leq x \leq \frac{\pi}{4}\}$ is the same as $M \cap \{(x,y): -\frac{\pi}{4} \leq x \leq \frac{\pi}{4}\}$ and $\partial S \cap \{(x,y): -\frac{\pi}{4} \leq x \leq \frac{\pi}{4}\}$ is the same as $\partial M \cap \{(x,y): -\frac{\pi}{4} \leq x \leq \frac{\pi}{4}\}$. Hence the conclusion is clear from the formula for L. $\square$

Hence we also have that $\Pi(\partial S) \cap \{(r,\theta): 1 \leq r \leq 3\frac{1}{20}$ and $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}\}$ is homeomorphic to the $\sin\frac{1}{x}$ continuum. Now the same argument can be used to show that $\partial S \cap \{(x,y): -\frac{5\pi}{4} \leq x \leq -\frac{3\pi}{4}\}$ is also homeomorphic to the $\sin\frac{1}{x}$ continuum. Since $\partial S \cap (\{-\pi\} \times [-\frac{1}{20}, 2]) = \{-\pi\} \times [\frac{1}{20}, \frac{11}{20}]$, $h\big(\partial S \cap (\{-\pi\} \times [-\frac{1}{20}, 2])\big) = \{0\} \times [0, \frac{1}{2}]$. Since $h(\bar{S}) \subset \bar{S}$, $h(\bar{S}) \cap ([-\frac{\pi}{4}, \frac{\pi}{4}] \times [-\frac{1}{20}, 2]) \subseteq \bar{S} \cap ([-\frac{\pi}{4}, \frac{\pi}{4}] \times [-\frac{1}{20}, 2])$ and each of $\partial S \cap ([-\frac{\pi}{4}, \frac{\pi}{4}] \times [-\frac{1}{20}, 2])$ and $h(\partial S) \cap ([-\frac{\pi}{4}, \frac{\pi}{4}] \times [-\frac{1}{20}, 2]) = \partial h(S) \cap ([-\frac{\pi}{4}, \frac{\pi}{4}] \times [-\frac{1}{20}, 2])$ is a $\sin\frac{1}{x}$ continuum with limit set $\{0\} \times [0, \frac{1}{2}]$. Therefore there is no simple curve which crosses from $x = -\frac{\pi}{4}$ to $x = \frac{\pi}{4}$ in $\bar{S} \setminus h(S)$. This implies, for this particular homeomorphism h and continuous function δ, that there is no simple curve in $\bar{S} \setminus h(S)$ which separates $y = -\frac{1}{20}$ from $y = 2$. So that there is no simple closed curve in $\pi(\bar{S}) \setminus g(\Pi(S))$ which separates $r = 1$ from $r = 3\frac{1}{2}$.

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